SOME MORE PATTERNS FROM PASCAL'S TRIANGLE

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1. INTRODUCTION

Over the years, much use has been made of Pascal's triangle, part of which is shown in Table 1.1. The original intention was to read the table horizontally, when its nth row gives, in order, the coefficients of x^m $\{m = 0, 1, \ldots, n\}$ for the binomial expansion of $(1 + x)^n$.

Pargeter [1] pointed out that the consecutive elements, read downwards, in the nth column gave the coefficients of x^m $\{m = 0, 1, \ldots, 00\}$ for the infinite expansion of $(1 - x)^n$. More recently, Fletcher [2] has considered the series whose coefficients are obtained (in the representation of Table 1.1) by starting on one of the diagonal unities and making consecutive "knight's moves" of two steps down and one to the right. Again, moving down the diagonals of Table 1.1, we obtain consecutive series of the so-called "figurative numbers," for instance, see Beiler [3]; and the ingenious reader will be able to find other interesting series, which can be simply generated. As with all work on integer sequences, Sloane [4] will be found invaluable.

Table 1.1 Pascal's Triangle

1											
1	1										
1	2	1									
1	3	3	1								
1	4	6	4	1							
1	5	10	10	5	1						
1	6	15	20	15	6	1					
1	7	21	35	35	21	7	1				
1	8	28	56	70	56	28	8	1			
1	9	36	84	126	126	84	36	9	1		
1	10	45	120	210	252	210	120	45	10	1	
1	11	55	165	330	462	462	330	165	55	11	1
⋮											⋮

However, the object of this paper is to draw attention to some equally striking, but rather more subtle patterns, obtainable from Pascal's triangle. The results emerged from the study of the determinants of a class of matrices which occurred naturally in a piece of statistical research, as reported by Anderson [5].

2. THE DETERMINANTAL VALUE FOR A FAMILY OF MATRICES

Consider any two positive integers r and s, and define the rth order determinant $D_r(s)$ as having $\binom{2s}{s+i-j}$ for its general i,jth element. Then it is well-known that

$$D_r(1) \equiv r + 1, \quad r \geq 1$$

and, interestingly enough, it can be shown that $D\ (s)$

$$\equiv \frac{(r+1)(r+2)^2 \cdots (r+s-1)^{s-1}(r+s)^s(r+s+1)^{s-1} \cdots (r+2s-2)^2(r+2s-1)}{1.2^2 \cdots (s-1)^{s-1}s^s(s+1)^{s-1} \cdots (2s-2)^2(2s-1)},$$

$r,\ s \geq 1$. For instance, see Anderson [6].

If we write out the family of determinants $D_r(s)$: $r,\ s \geq 1$ as a doubly infinite two-dimensional array, we get Table 2.1.

Table 2.1 Values for Family of Determinants $D_r(s)$

		s = 1	2	3	4	$\cdots$
r	1	2	6	20	70	$\cdots$
	2	3	20	175	1764	
	3	4	50	980	24696	
	4	5	105	4116	232848	$\cdots$
	$\vdots$	$\vdots$			$\vdots$	$\ddots$

What is really pretty is that this table can be written down quite simply, and in several ways, from Pascal's triangle.

3. GENERATING THE DETERMINANTS FROM PASCAL'S TRIANGLE

3.1 Generating the Rows of Table 2.1

The first row is obtained just by making the "knight's moves" from the apex of the triangle—see Table 3.1.1. If we then evaluate the second-order determinants, with leading terms at these "knight's moves," we obtain the second row—see Table 3.1.2. Similarly, if we evaluate the third-order determinants, shown in Table 3.1.3, and the fourth-order determinanys, shown in Table 3.1.4, we get the third and fourth rows, respectively. In both cases, the determinants have the "knight's moves" for their leading terms. Continuing in this way, Table 2.1 can be extended to as many rows as we like.

Table 3.1.1 Generation of the First Row

$$\begin{tikzcd} 2 \arrow[d] & & & \\ {} \arrow[r] & 6 \arrow[d] & & \\ & {} \arrow[r] & 20 \arrow[d] & \\ & & {} \arrow[r] & 70 \\ & & & \ddots \end{tikzcd}$$

Table 3.1.2 Generation of the Second Row

$$\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} \quad \begin{vmatrix} 6 & 4 \\ 10 & 10 \end{vmatrix} \quad \begin{vmatrix} 20 & 15 \\ 35 & 35 \end{vmatrix} \quad \begin{vmatrix} 70 & 56 \\ 126 & 126 \end{vmatrix} \quad \ddots$$

Table 3.1.3 Generation of the Third Row

$$\begin{vmatrix} 2 & 1 & 0 \\ 3 & 3 & 1 \\ 4 & 6 & 4 \end{vmatrix} \quad \begin{vmatrix} 6 & 4 & 1 \\ 10 & 10 & 5 \\ 15 & 20 & 15 \end{vmatrix} \quad \begin{vmatrix} 20 & 15 & 6 \\ 35 & 35 & 21 \\ 56 & 70 & 56 \end{vmatrix} \quad \begin{vmatrix} 70 & 56 & 28 \\ 126 & 126 & 84 \\ 210 & 252 & 210 \end{vmatrix} \quad \ddots$$

Table 3.1.4 Generation of the Fourth Row

2	1	0	0				
3	3	1	0				
4	6	4	1	0			
5	10	10	5	1			
	15	20	15	6	1		
	21	35	35	21	7		
		56	70	56	28	8	
		84	126	126	84	36	
			210	252	210	120	
			330	462	462	330	
							⋱

3.2 Generating the Columns of Table 2.1

The first column can be picked out as the natural number diagonal of Table 2.1, shown in Table 3.2.1. For the second column, the overlapping second-order determinants, shown in Table 3.2.2, are evaluated; while the third and fourth columns are obtained from the determinants of the overlapping arrays in Tables 3.2.3 and 3.2.4, respectively. And so on.

Table 3.2.1 Generation of the First Column

2				
	3			
		4		
			5	
				⋱

Table 3.2.2 Generation of the Second Column

3	3				
4	6	4			
	10	10	5		
		20	15	6	
			35	21	
					⋱

Table 3.2.3 Generation of the Third Column

4	6	4				
5	10	10	5			
6	15	20	15	6		
	21	35	35	21	7	
		56	70	56	28	
			126	126	84	
						⋱

Table 3.2.4 Generation of the Fourth Column

5	10	10	5				
6	15	20	15	6			
7	21	35	35	21	7		
8	28	56	70	56	28	8	
	36	84	126	126	84	36	
		120	210	252	210	120	
			330	462	462	330	
							⋱

3.3 Generating the Diagonals of Table 2.1

Finally, the diagonals of Table 2.1 can be obtained as follows. The tth member of the main diagonal is found by evaluating the tth-order determinant with leading terms given by the tth "knight's move" from the apex of the triangle. The first super-diagonal is achieved using the same principle, but starting with the second "knight's move." Thus its tth term is the tth-order determinant whose leading term is the $(t+1)$th "knight's move." The tth term of the second super-diagonal is given by the tth-order determinant, starting with the $(t+2)$th "knight's move." And so on for all the other super-diagonals. The sub-diagonals can be obtained in a similar way; by using, instead of the "knight's move" sequence, a sequence diagonally down from it in the triangle. For the first sub-diagonal, the new sequence is one step down; for the second, two steps down, and so on.

4. GENERATING THE DETERMINANTS FROM A DIFFERENT REPRESENTATION OF PASCAL'S TRIANGLE

If we represent Pascal's triangle as in Table 4.1, where the ones have been omitted, we get a more meaningful row and column array. We then find that Table 2.1 can be generated in still further ways, as the reader can readily verify.

TABLE 4.1 Alternative Representation of Pascal's Triangle

2	3	4	5	6	...
3	6	10	15	21	
4	10	20	35	56	
5	15	35	70	126	
6	21	56	126	252	
⋮				⋮	⋱

5. IN CONCLUSION

All the patterns discussed can, of course, be verified by combinatorial algebra. Thus, for instance, in Section 3, the second column of Table 2.1 is claimed to have for its nth element the second-order determinant:

$$\begin{vmatrix} \binom{n+1}{n-1} & \binom{n+1}{n} \\ \\ \binom{n+2}{n-1} & \binom{n+2}{n} \end{vmatrix}.$$

On evaluation, this gives

$$\frac{(n+1)(n+2)^2(n+3)}{12}$$

as required.

REFERENCES

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4. N. J. A. Sloane, *A Handbook of Integer Sequences* (New York & London: Academic Press, 1973).
5. O. D. Anderson, "A Note on Differencing Autoregressive Moving Average (p, q) Processes," *Biometrika*, Vol. 62, No. 2 (December 1975), pp. 521-523.
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