AN INVARIANT FOR COMBINATORIAL IDENTITIES

LEON BERNSTEIN
Illinois Institute of Technology, Chicago, IL 60616

Dedicated to Dr. Thomas L. Martin, Jr.

0. INTRODUCTION

By an invariant of a mathematical structure—a matrix, an equation, a field—we usually understand a relation, or a formula emerging from that structure—which remains unaltered if certain operations are performed on this structure. An invariant is, so to speak, the calling card of some mathematical pattern, it is a fixed focus around which the infinite elements of this pattern revolves. Matrices, the general quadratic, and many other mathematical configurations have their invariants. So do groups, if they are not simple. A prima donna invariant is the class number of algebraic number fields. She is far from having been unveiled. Some serenades have been sung to her from the quadratic, and to a much lesser extent, the cubic fields. Higher fields are absolutely taboo for their class number, and will probably remain so for many decades to come. With certain restrictions, also the set of fundamental units of an algebraic number field is an invariant.

This paper states a new invariant for all cubic fields. In a further paper a similar invariant will be stated for all algebraic number fields of any degree. Here the cubic case is singled out, and completely solved, since the technique, used in this paper, will carry over, step by step, to the general case. We shall outline the idea of this new invariant, as obtained here in the cubic case. Let e be any unit (not necessarily a fundamental one) of a cubic number field. Since e and e^{-1} are of third degree, both can be used as bases for the field. This must not be a minimal basis, so that we can put

$$e^v = x_v + y_v e + z_v e^2,\ x_v, y_v, z_v \in \mathbb{Z},\ v = 0, 1, \ldots,\ e^{-v} = r_v + s_v e^{-1} + t_v e^{-2}.$$

x_v and r_v are then calculated explicitly as arithmetic functions of v. From $e^v \cdot e^{-v} = 1$, we obtain the combinatorial identity

$$x_v = r_v^2 - r_{v-1} r_{v+1},$$

and this is an invariant, regardless of how the cubic field and one of its units is chosen. We also obtain a second invariant, viz.,

$$r_v = x_v^2 - x_{v-1} x_{v+1}.$$

Few invariants can please better the heart of a mathematician.

1. POWERS OF UNITS

Let

$$F(x) = x^3 + c_1x^2 + c_2x + c_3;\ c_1, c_2, c_3 \in \mathbb{Z} \tag{1.1}$$

be an irreducible polynomial in x over $\mathbb{Z}$ of negative discriminant, having one real root w, and one pair of conjugate roots. By Dirichlet's theorem, $Q(w)$ has exactly one fundamental unit e, viz.,

$$e = r_1 + r_2w + r_3w^2;\ r_1, r_2, r_3 \in Q.$$

Of course, e is a third-degree algebraic irrational. Since

$$0 = w^3 + c_1w^2 + c_2w + c_3,$$

we find the field equation of e by the known method

$$\begin{aligned} e &= r_1 + r_2w + r_3w^2, \\ we &= r_1' + r_2'w + r_3'w^2,\ (r_1',\ r_2',\ r_3' \in Q) \\ w^2e &= r_1'' + r_2''w + r_3''w^2,\ (r_1'',\ r_2'',\ r_3'' \in Q) \end{aligned}$$

and obtain

$$\begin{cases} e^3 - a_1e^2 - a_2e - a_3 = 0, \\ a_1,\ a_2,\ a_3 \in \mathbb{Z},\ a_3 = \pm 1. \end{cases} \tag{1.2}$$

Here we investigate, w.e.g., the case $a_3 = 1$, hence

$$\begin{aligned} &e^3 - a_1e^2 - a_2e - 1 = 0 \\ &e^3 = 1 + a_2e + a_1e^2;\ a_1,\ a_2 \neq 0, \text{ by presumption.} \end{aligned} \tag{1.2a}$$

Our further aim is to obtain explicit expressions for the positive and negative powers of e. To achieve this, we take refuge to a very convenient trick which makes the calculations uncomparably easier. We use as a basis for $Q(w)$ the triples 1, e, e^2 and 1, e^{-1}, e^{-2}; the question whether these are minimal bases is not relevant here. We put

$$e^v = x_v + y_ve + z_ve^2;\quad x_v, y_v, z_v \in \mathbb{Z};\ v = 0, 1, \ldots, \tag{1.3}$$

$$x_0 = 1,\ x_1 = x_2 = 0. \tag{1.3a}$$

We obtain from (1.3), multiplying by e, and with (1.2a)

$$\begin{aligned} e^{v+1} &= x_ve + y_ve^2 + z_v(1 + a_2e + a_1e^2) \\ &= z_v + (x_v + a_2z_v)e + (y_v + a_1z_v)e^2 \\ &= x_{v+1} + y_{v+1}e + z_{v+1}e^2, \end{aligned}$$

Hence, by comparison of coefficients,

$$\begin{aligned} x_{v+1} &= z_v, \\ y_{v+1} &= x_v + a_2 z_v, \\ z_{v+1} &= y_v + a_1 z_v. \end{aligned}$$

Thus, we obtain

$$\begin{cases} y_v = x_{v-1} + a_2 x_v; \quad (v = 1, 2, \ldots) \\ z_v = x_{v+1} \\ e^v = x_v + (x_{v-1} + a_2 x_v)e + x_{v+1}e^2 \end{cases} \tag{1.4}$$

$$x_{v+2} = x_{v-1} + a_2 x_v + a_1 x_{v+1};$$

$$x_{v+3} = x_v + a_2 x_{v+1} + a_1 x_{v+2}; \quad (v = 0, 1, \ldots). \tag{1.4a}$$

Formula (1.4a) is the recurrence relation which will enable us to calculate explicitly x_v, and with it e^v. We set

$$\sum_{v=0}^{\infty} x_v u^v = x_0 + x_1 u + x_2 u^2 + \sum_{v=3}^{\infty} x_v u^v,$$

and, with the initial values from (1.3a),

$$\sum_{v=0}^{\infty} x_v u^v = 1 + \sum_{v=3}^{\infty} x_v u^v = 1 + \sum_{v=0}^{\infty} x_{v+3} u^{v+3}. \tag{1.4b}$$

Substituting on the right side the value of x from (1.4a) [and taking into account (1.3a)], we obtain

$$\begin{aligned} \sum_{v=0}^{\infty} x_v u^v &= 1 + \sum_{v=0}^{\infty} (x_v + a_2 x_{v+1} + a_1 x_{v+2}) u^{v+3} \\ &= 1 + u^3 \sum_{v=0}^{\infty} x_v u^v + a_2 u^2 \sum_{v=0}^{\infty} x_{v+1} u^{v+1} + a_1 u \sum x_{v+2} u^{v+2} \\ &= 1 + u^3 \sum_{v=0}^{\infty} x_v u^v + a_2 u^2 \left[\left(\sum_{v=0}^{\infty} x_v u^v\right) - x_0\right] + a_1 u \left[\left(\sum_{v=0}^{\infty} x_v u^v\right) - x_0 - x_1 u\right] \\ &= 1 + (u^3 + a_2 u^2 + a_1 u) \sum_{v=0}^{\infty} x_v u^v - a_2 u^2 - a_1 u. \end{aligned}$$

We have thus obtained

$$(1 - a_1 u - a_2 u^2 - u^3) \sum_{v=0}^{\infty} x_v u^v = 1 - a_1 u - a_2 u^2. \tag{1.4c}$$

Since u is an indeterminate, and can assume any value, we choose

$$1 - a_1 u - a_2 u^2 - u^3 \neq 0, \tag{1.4d}$$

and obtain from (1.4c) and (1.4d)

$$\sum_{v=0}^{\infty} x_v u^v - \frac{1 - a_1 u - a_2 u^2}{1 - a_1 u - a_2 u^2 - u^3},$$

$$\sum_{v=0}^{\infty} x_v u^v = 1 + \frac{u^3}{1 - a_1 u - a_2 u^2 - u^3},$$

and from (1.4b)

$$\sum_{v=0}^{\infty} x_v u^v = 1 + \sum_{v=0}^{\infty} x_{v+3} u^{v+3} = 1 + \frac{u^3}{1 - a_1 u - a_2 u^2 - u^3},$$

$$\sum_{v=0}^{\infty} x_{v+3} u^{v+3} = \frac{u^3}{1 - a_1 u - a_2 u^2 - u^3},$$

and since $u \neq 0$,

$$\sum_{v=0}^{\infty} x_{v+3} u^v = \frac{1}{1 - u(a_1 + a_2 u + u^2)}. \tag{1.4e}$$

Choosing, additionally to (1.4d),

$$0 < |u(a_1 + a_2 u + u^2)| < 1,$$

we obtain, from (1.4d)

$$\sum_{v=0}^{\infty} x_{v+3} u^v = \sum_{j=0}^{\infty} u^j (a_1 + a_2 u + u^2)^j. \tag{1.5}$$

To calculate x_v explicitly, we shall compare the coefficients of u^m ($m = 0, 1, \ldots$) on each side of (1.5). On the left, this equals to x_{m+3}. On the right side we investigate

$$\left\{\begin{aligned} &\sum_{i=0} u^{m-i}(a_1 + a_2 u + u^2)^{m-i} = \\ &\sum_{i=0}^{m-i} u^{m-i} \sum_{y_1 + y_2 + y_3 = m-i} \binom{m-i}{y_1, y_2, y_3} a_1^{y_1} (a_2 u)^{y_2} u^{2y_3}. \end{aligned}\right. \tag{1.5a}$$

Since we demand that the element u have the exponent m, we obtain

$$m - i + y_2 + 2y_3 = m.$$

$$y_2 + 2y_3 = i, \tag{1.5b}$$

$$y_2 = i - 2y_3, \tag{1.5c}$$

and

$$y_1 + y_2 + y_3 = m - i \tag{1.5d}$$

yield

$$y_1 = m - i - i + 2y_3 - y_3,$$

$$y_1 = m - 2i + y_3. \tag{1.5e}$$

We further have

$$\binom{m - i}{y_1, y_2, y_3} = \frac{(m - i)!}{y_1!y_2!y_3!} = \frac{(m - i)!}{(m - 2i + y_3)!(i - 2y_3)!y_3!}$$

$$= \frac{(m - i)!(i - y_3)!}{(i - y_3)!(m - 2i + y_3)!(y - 2y_3)!y_3!}$$

$$= \binom{m - i}{i - y_3}\binom{i - y_3}{y_3},$$

$$\binom{m - i}{y_1, y_2, y_3} = \binom{m - i}{i - y_3}\binom{i - y_3}{y_3}. \tag{1.5f}$$

Writing j for y_3, we thus obtain

$$x_{m+3} = \sum_{i=0}\sum_{j=0}\binom{m - i}{i - j}\binom{i - j}{j}a_1^{m-2i+j}a_2^{i-2j}. \tag{1.5g}$$

We shall determine the upper bounds of i and j. From the binomial coefficient $\binom{i - j}{j}$, we obtain

$$j \le i - j, \quad 2j \le i, \quad j \le \frac{i}{2}; \tag{1.5h}$$

From the binomial coefficient $\binom{m - i}{i - j}$, we obtain

$$m - i \ge i - j, \quad m - 2i \ge -j,$$

and from (1.5h), $-j \ge -\frac{i}{2}$, so that

$$m - 2i \ge -\frac{i}{2}, \quad m \ge \frac{3}{2}i, \quad i \le \frac{2}{3}m. \tag{1.5i}$$

From (1.5h) and (1.5i), we have thus obtained

$$i \le \left[\frac{2m}{3}\right]; \quad j \le \left[\frac{i}{2}\right],$$

hence,

$$x_{m+3} = \sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-i}{i-j}\binom{i-j}{j} a_1^{m-2i+j} a_2^{i-2j}; \quad (m = 0, 1, \ldots) \tag{1.6}$$

We shall verify formula (1.6) which does not lack harmony in its simple structure. From (1.3a) and (1.4a), we obtain, for $v = 0, 1, \ldots,$

$$\begin{aligned} x_3 &= 1, \\ x_4 &= a_1, \\ x_5 &= a_2 + a_1^2, \\ x_6 &= 1 + 2a_1a_2 + a_1^3. \end{aligned}$$

From (1.6), we obtain, for $m = 0, 1, 2, 3,$

$m = 0$, $x_3 = 1$, since $i = j = 0$, $\binom{0}{0} \stackrel{\text{def}}{=} 1$.

$m = 1$; $i = j = 0$, $x_4 = a_1$;

$m = 2$; $i = 0$, $j = 0$; $i = 1$, $j = 0$, $x_5 = a_1^2 + a_2$;

$m = 3$; $i = 0$, $j = 0$; $i = 1$, $j = 0$; $i = 2$, $j = 1$, $x_6 = a_1^2 + 2a_1a_2 + 1$.

We shall proceed to calculate the negative powers of e, and put

$$e^{-v} = r_v + s_v e^{-1} + t_v e^{-2}. \tag{1.7}$$

For the initial values, we obtain again

$$v = 0, 1, 2; \quad r_0 = 1; \quad r_1 = r_2 = 0. \tag{1.8}$$

For the field equation of e^{-1}, we obtain, from (1.2a),

$$e^{-3} = 1 - a_1e^{-1} - a_2e^{-2}; \; a_1, \, a_2 \neq 0. \tag{1.9}$$

If we compare (1.9) with (1.2a), we see that the recursion formula for e^{-v}, with the same initial values for $v = 0, 1, 2$, is the same as that for e^v, substituting only $-a_1$ for a_2 and $-a_2$ for a_1; hence we obtain, in complete analogy with (1.4), (1.4a), and (1.6),

$$\left\{\begin{aligned} s_v &= r_{v-1} - a_1r_v, \\ t_v &= r_{v+1} \\ e^{-v} &= r_v + (r_{v-1} - a_1r_v)e^{-1} + r_{v+1}e^{-2} \\ r_{v+3} &= r_v - a_1r_{v+1} - a_2r_{v+2} \end{aligned}\right. \tag{1.9a}$$

$$r_{m+3} = \sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-i}{i-j}\binom{i-j}{j} (-a_2)^{m-2i+j}(-a_1)^{i-2j}$$

$$r_{m+3} = \sum_{i=0}^{[\frac{2m}{3}]} \sum_{j=0}^{[\frac{i}{2}]} (-1)^{m-i-j} \binom{m-i}{i-j} \binom{i-j}{j} a^{i-2j} a^{m-2i+j}; \quad (m = 0, 1, \ldots). \tag{1.9b}$$

Formulas (1.6) and (1.9b) are our main tools in establishing new identities of combinatorial structures. Both x_m and r_m are arithmetic functions, and we shall show that there exist simple relations between them.

2. TRUNCATED FIELD EQUATIONS OF UNITS

We shall now drop the restriction (1.2a), viz., $a_1, a_2 \neq 0$, and investigate the cases when either a_1 or a_2 equal zero. We shall start with

$$a_2 = 0, \; e^3 = 1 + a_1 e^2; \quad a_1 \neq 0. \tag{2.1}$$

$$(e \text{ a cubic unit; } a_1 \in \mathbb{Z})$$

Formulas (1.9) take the form, setting

$$e^v = x_v + y_v e + z_v e^2 \; (v = 0, 1, \ldots; \; x_v, y_v, z_v \in \mathbb{Z}) \tag{2.2}$$

$$\begin{cases} y_v = x_{v-1}, \\ z_v = x_{v+1}, \\ e^v = x_v + x_{v-1}e + x_{v+1}e^2, \; (v = 1, 2, \ldots) \end{cases} \tag{2.2a}$$

and (1.4a) becomes

$$x_{v+2} = x_v + a_1 x_{v+3}, \; (x_0 = 1, \; x_1 = x_2 = 0; \; v = 0, 1, \ldots). \tag{2.3}$$

To calculate x_v explicitly from (2.3), there is no need to go through the whole process of using Euler's generating functions. Instead, we can proceed straight to formula (1.5a). Here we shall then keep in mind though, that the condition $a_2 = 0$ results in $y_2 = 0$, and we obtain

$$\sum_{i=0} u^{m-i}(a_1 + u)^{m-i} = \sum_{i=0} u^{m-i} \sum_{j=0} \binom{m-i}{j} a_1^{m-i-j} u^{2j} \; . \tag{2.4}$$

Since we are looking for powers u^m, we obtain

$$m - i + 2j = m,$$

$$i = 2j; \tag{2.4a}$$

since, from the binomial coefficient on the right side of (2.4),

$$m - i \geq j, \quad m - 2j \geq j, \quad j \leq \frac{m}{3} \tag{2.4b}$$

and formula (1.6) takes here the final form,

$$x_{m+3} = \sum_{j=0}^{[\frac{m}{3}]} \binom{m-2j}{j} a_1^{m-3j}, \quad m = 0, 1, \ldots . \tag{2.5}$$

Instead of proceeding to calculate the negative powers of e for the case $a_2 = 0$ from $e^3 = 1 + a_1 e^2$, we shall first calculate the positive powers of e for the case $a_1 = 0$. The reasons for this will become clear in the sequel. We set again

$$\begin{cases} e^v = x_v + y_v e + z_v e^2; \; x_0 = 1; \; x_1 = x_2 = 0 \\ e^3 = 1 + a_2 e; \; y_v = x_{v-1} + a_2 x_v; \; z_v = x_{v+1}; \; x_{3+v} = x_v + a_2 x_{v+1}; \; a_2 \neq 0. \end{cases} \tag{2.6}$$

(1.5) now takes the form

$$\sum_{v=0}^{\infty} x_{v+3} u^v = \sum_{j=0}^{\infty} u^{2j} (a_2 + u)^j . \tag{2.6a}$$

It is convenient to calculate x_{2m+3} and x_{2m+4} separately because of the factor u^{2j} under the second sigma sign. Because of the factor u^{2j}, we shall calculate separately the coefficients of u^{2m} $(v = 2m)$ and u^{2m+1} $(v = 2m+1)$. We obtain, after easy calculations,

$$\begin{cases} x_{2m+3} = \displaystyle\sum_{i=0}^{[\frac{m}{3}]} \binom{m-i}{2i} a_2^{m-3i}, & (m = 0, 1, \ldots) \\ x_{2m+4} = \displaystyle\sum_{i=0}^{[\frac{m-i}{3}]} \binom{m-i}{2i+1} a_2^{m-3i-1}, & (m = 1, 2, \ldots) \\ (x_4 = 0). \end{cases} \tag{2.6b}$$

We can now easily calculate the negative powers for e^v in the cases $a_1 = 0$, and $a_2 = 0$. In the case $a_1 = 0$, we obtain, from (2.6)

$$\begin{cases} e^{-v} = r_v + s_v e^{-1} + t_v e^{-2}, \\ e^{-3} = 1 - a_2 e^{-2}, \end{cases} \tag{2.7}$$

and from (2.2a),

$$e^{-v} = r_v + r_{v-1} e^{-1} + r_{v+1} e^{-2}, \tag{2.7a}$$

and from (2.5),

$$r_{v+3} = \sum_{j=0}^{[\frac{m}{3}]} \binom{m-2j}{j} (-a_2)^{m-3j}$$

$$r_{v+3} = \sum_{j=0}^{[\frac{m}{3}]} \binom{m-2j}{j}(-1)^{m-j}a_2^{m-3j}, \quad m = 0, 1, \ldots . \tag{2.7b}$$

In the case of $a_2 = 0$, we obtain, from (2.1),

$$\begin{cases} e^{-v} = r_v + s_v e^{-1} + t_v e^{-2} \\ e^{-3} = 1 - a_1 e^{-1}, \end{cases} \tag{2.7c}$$

and from (2.6),

$$e^{-v} = r_v + (r_{v-1} - a_1 r_v)e^{-1} + r_{v+1}e^{-2},$$

and from (2.6b)

$$r_{2m+3} = \sum_{i=0}^{[\frac{m}{3}]} \binom{m-i}{2i}(-a_1)^{m-3i}, \qquad (m = 0, 1, \ldots)$$

$$\underline{r_{2m+3} = \sum_{i=0}^{[\frac{m}{3}]} (-1)^{m-i}\binom{m-i}{2i}a_1^{m-3i}}, \qquad (m = 0, 1, \ldots) \tag{2.7d}$$

$$\underline{r_{2m+4} = \sum_{i=0}^{[\frac{m-i}{3}]} (-1)^{m-i-1}\binom{m-i}{2i+1}a_1^{m-3i-1}}, \quad (m = 1, 2, \ldots). \tag{2.7e}$$

3. COMBINATORIAL IDENTITIES

In this section, we shall establish the new combinatorial identities, by means of the powers of the units which we have stated explicitly in §1. We shall enumerate the main results we have obtained there in order to save the reader unnecessary backpaging.

$$\begin{cases} e^3 = 1 + a_2 e + a_1 e^2;\ a_1,\ a_2 \neq 0 \text{ (by presumption)}; \\ e^v = x_v + y_v e + z_v e^2;\ x_v, y_v, z_v, a_1, a_2 \in \mathbb{Z}; \\ y_v = x_{v-1} + a_2 x_v;\ z_v = x_{v+1}; \\ x_{v+3} = x_v + a_2 x_v \quad + a_1 x_v \quad ;\ x_0 = 1;\ x_1 = x_2 = 0; \\ e^{-v} = r_v + s_v e^{-1} + t_v e^{-2};\ r_v, s_v, t_v \in \mathbb{Z}; \\ s_v = r_{v-1} - a_1 r_v;\ t_v = r_{v+1}; \\ r_{v+3} = r_v - a_1 r_{v+1} - a_2 r_{v+2};\ r_0 = 1,\ r_1 = r_2 = 0; \\ e^{-3} = 1 - a_1 e^{-1} - a_2 e^{-2}. \end{cases} \tag{3.1}$$

From the last equation of (3.1), multiplying both sides first by e^2, then by e^{-1}, we obtain

$$\begin{aligned} e^{-1} &= -a_2 - a_1 e + e^2, \\ e^{-2} &= -a_2(-a_2 - a_1 e + e^2) - a_1 + e, \\ e^{-1} &= -a_2 - a_1 e + e^2;\ e \quad = a_2^2 - a_1(a_1 a_2 + 1)e - a_2 e^2. \end{aligned} \tag{3.2}$$

We now obtain, from (3.1) and (3.2),

$$\left\{\begin{aligned} 1 = e^v e^{-v} &= (x_v + y_v e + z_v e^2)(r_v + s_v e^{-1} + t_v e^{-2}) \\ &= x_v r_v + y_v s_v + z_v t_v + (y_v r_v + z_v s_v)e + z_v r_v e^2 \\ &\qquad + (x_v s_v + y_v t_v)e^{-1} + x_v t_1 e^{-2} \\ &= x_v r_v + y_v s_v + z_v t_v + (y_v r_v + z_v s_v)e + z_v r_v e^2 \\ &\qquad + (x_v s_v + y_v t_v)(-a_2 - a_1 e + e^2) \\ &\qquad + [a_2^2 - a_1 + (a_1 a_2 + 1)e - a_2 e^2]x_v t_v. \end{aligned}\right. \tag{3.2a}$$

$$\left\{\begin{aligned} 1 = x_v r_v &+ (y_v - a_2 x_v)s_v + [z_v + (a_2^2 - a_1)x_v - a_2 y_v]t_v \\ &+ \big(y_v r_v + (z_v - a_1 x_v)s_v + [(a_1 a_2 + 1)x_v - a_1 y_v]t_v\big)e \\ &+ \big((z_v r_v + x_v s_v + (y_v - a_2 x_v)t_v\big)e^2. \end{aligned}\right. \tag{3.2b}$$

Comparing in (3.2b) coefficients of equal powers of e on both sides, and reminding that e is a cubic irrational, we obtain the system of three linear equations in the three indeterminates r_v, s_v, and t_v,

$$\left\{\begin{aligned} & x_v r_v + (y_v - a_2 x_v)s_v + [z_v + (a_2^2 - a_1)x_v - a_2 y_v]t_v = 1, \\ & y_v r_v + (z_v - a_1 x_v)s_v + [(a_1 a_2 + 1)x_v - a_1 y_v]t_v = 0, \\ & z_v r_v + x_v s_v + (y_v - a_2 x_v)t_v = 0. \end{aligned}\right. \tag{3.3}$$

Adding to the first equation of (3.3) the a_2 multiple of the third one, we obtain, adding also to the second the a_1 multiple of the third,

$$\left\{\begin{aligned} & (x_v + a_2 z_v)r_v + y_v s_v + (z_v - a_1 x_v)t_v = 1, \\ & (y_v + a_1 z_v)r_v + z_v s_v + x_v t_v = 0, \\ & z_v r_v + x_v s_v + (y_v - a_2 x_v)t_v = 0. \end{aligned}\right. \tag{3.3a}$$

Since the indeterminates r_v, s_v, t_v are to be expressed by x_v, y_v, z_v, we calculate the determinant Δ_{v+2} of the system (3.3a), viz.,

$$\begin{vmatrix} x_v + a_2 z_v & y_v & z_v - a_1 x_v \\ y_v + a_1 z_v & z_v & x_v \\ z_v & x_v & y_v - a_2 x_v \end{vmatrix} = \Delta_{v+2} \tag{3.3b}$$

Why this determinant has the index $v + 2$, and not v, as would seem proper, will be understood, and justified, soon.

We have, for the first row of the determinant (3.3b) from (3.1), and similarly for the second and third

$$\begin{cases} x_v + a_2 z_v = x_v + a_2 x_{v+1}, \\ y_v = x_{v-1} + a_2 x_v, \\ z_v - a_1 x = x_{v+1} - a_1 x_v = x_{v-2} + a_2 x_{v-1}. \end{cases} \tag{3.3c}$$

With (3.3c), (3.3b) becomes

$$\Delta_{v+2} = \begin{vmatrix} x_v + a_2 x_{v+1} & x_{v-1} + a_2 x_v & x_{v-2} + a_2 x_{v-2} \\ x_{v+2} & x_{v+1} & x_v \\ x_{v+1} & x_v & x_{v-1} \end{vmatrix}. \tag{3.3d}$$

The third row of (3.3d) is obtained from (3.1) as follows:

$$z_v = x_{v+1};\; y_v - a_2 x_v = x_{v-1} + a_2 x_v - a_2 x_v = x_{v-1};$$

the first entry of the second row is obtained as follows:

$$y_v + a_1 x_v = x_{v-1} + a_2 x_v + a_1 x_{v+1} = x_{v+2}.$$

Subtracting in the determinant of (3.3d) from the first row the a_2-multiple of the third row, we obtain

$$\Delta_{v+2} = \begin{vmatrix} x_v & x_{v-1} & x_{v-2} \\ x_{v+2} & x_{v+1} & x_v \\ x_{v+1} & x_v & x_{v-1} \end{vmatrix}. \tag{3.3e}$$

Interchanging in (3.3e) the first row with the second, and then the second with the third, we finally obtain

$$\Delta_{v+2} = \begin{vmatrix} x_{v+2} & x_{v+1} & x_v \\ x_{v+1} & x_v & x_{v-1} \\ x_v & x_{v-1} & x_{v-2} \end{vmatrix}. \tag{3.3f}$$

Substituting for the entries of the first row of (3.3f) the value from (3.1), viz.,

$$x_{k+3} = x_k + a_2 x_{k+1} + a_1 x_{k+2},\; (k = k+2,\; v+1,\; v)$$

$$\Delta_{v+2} = \begin{vmatrix} x_{v-1} + a_2 x_v + a_1 x_{v+1} & x_{v-2} + a_2 x_{v-1} + a_1 x_v & x_{v-3} + a_2 x_{v-2} + a_1 x_{v-1} \\ x_{v+1} & x_v & x_{v-1} \\ x_v & x_{v-1} & x_{v-2} \end{vmatrix}. \tag{3.3g}$$

Subtracting in (3.3g) from the first row the a_1-multiple of the second, and the a_2-multiple of the third, we obtain

$$\Delta_{v+2} = \begin{vmatrix} x_{v-1} & x_{v-2} & x_{v-3} \\ x_{v+1} & x_v & x_{v-1} \\ x_v & x_{v-1} & x_{v-2} \end{vmatrix}. \tag{3.3h}$$

Interchanging in (3.3h) the first row with the second, and then the second with the third, we obtain

$$\Delta_{v+2} = \begin{vmatrix} x_{v+1} & x_v & x_{v-1} \\ x_v & x_{v-1} & x_{v-2} \\ x_{v-1} & x_{v-2} & x_{v-3} \end{vmatrix}. \tag{3.3i}$$

From (3.3f) and (3.3i) we obtain the important result

$$\Delta_{v+2} = \Delta_{v+1} = \Delta_k; \ (k = 5, 6, \ldots). \tag{3.4}$$

Taking in (3.4) $k = 5$, and reminding, from (3.1), that $x_3 = 1$, $x_1 = x_2 = 0$, we obtain

$$\Delta_{v+2} = \begin{vmatrix} x_5 & x_4 & 1 \\ x_4 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1,$$

$$\Delta_{v+2} = -1. \tag{3.4a}$$

With (3.4a) we have finally calculated the determinant of the system of equations (3.3a). By Cramer's rule we now obtain from (3.3a) and (3.4a),

$$r_v = -\begin{vmatrix} z_v & x_v \\ x_v & y_v - a_2 x_v \end{vmatrix} \tag{3.5}$$

Substituting in (3.5),

$$z_v = x_{v+1}, \ y_v = x_{v-1} + a_2 x_v,$$

we obtain

$$\underline{\underline{r_v = x_v^2 - x_{v-1}x_{v+1}}}; \ (v = 1, 2, \ldots). \tag{3.6}$$

(3.6) is the desired combinatorial identity. Its full beauty will be appreciated when we substitute the values for r_v and x_v. Its simple structure in the form (3.6) is really astonishing. We must explain its remarkable origin. The reason for this harmoniousness is the fact that we have chosen to manipulate with the powers of a unit e in $Q(w)$ and a basis of the powers of e as the basis of $Q(w)$. For only this leads to the determinant Δ_{v+2} of the system of equations (3.3a) equal to ± 1. Had we chosen any other cubic irrational α in $Q(w)$, then the identity $\alpha^v \cdot \alpha^{-v} = 1$ would have led to a system of equations whose determinant is generally different from ±1. Formula (3.6) is therefore, in a certain sense, *an invariant of any cubic field* Q(w), *read of all the cubic fields*. The surprising explanation for this is the relationship

$$\Delta_{v+2} = N(e^v) = \big(N(e)\big)^v = \pm 1; \; (v = 0, 1, \ldots). \tag{3.7}$$

We shall prove (3.7). We obtain, denoting

$$\begin{aligned}
\alpha &= e^v = x_v + y_v e + z_v e^2; \; e^3 = 1 + a_2 e + a_1 e^2; \\
\alpha e &= x_v e + y_v e^2 + z_v(1 + a_2 e + a_1 e^2), \\
\alpha e &= z_v + (x_v + a_2 z_v)e + (y_v + a_1 z_v)e^2, \\
\alpha e^2 &= y_v + a_1 z_v + [a_2 y_v + (a_1 a_2 + 1)z_v]e + [x_v + a_1 y_v + (a_1^2 + a_2)z_v]e^2.
\end{aligned}$$

We thus obtain

$$\left\{\begin{array}{l} N(\alpha) = \\ (-1)^3 \begin{vmatrix} x_v & y_v & z_v \\ z_v & x_v + a_2 x_v & y_v + a_1 z_v \\ y_v + a_1 z_v & a_2 y_v + (a_1 a_2 + 1)z_v & x_v + a_1 y_v + (a_1^2 + a_2)z_v \end{vmatrix}. \end{array}\right. \tag{3.7a}$$

Subtracting in the determinant (3.7a) from the third row the a_1-multiple of the second row, we obtain

$$N(e^v) = -\begin{vmatrix} x_v & y_v & z_v \\ z_v & x_v + a_2 x_v & y_v + a_1 z_v \\ y_v & a_2 y_v - a_1 x_v + z_v & x_v + a_2 z_v \end{vmatrix},$$

and subtracting from the second column the a_2-multiple of the first column,

$$N(e^v) = -\begin{vmatrix} x_v & y_v - a_2 x_v & z_v \\ z_v & x_v & y_v + a_1 z_v \\ y_v & z_v - a_1 x_v & x_v + a_2 z_v \end{vmatrix}. \tag{3.7b}$$

Comparing (3.3b) with (3.7b), we obtain

$$N(e^v) = \Delta_{v+2} = 1. \tag{3.7c}$$

Had we chosen any $\alpha \in Q(w)$, formula (3.6) would take the form

$$N(\alpha) r_v = x_v^2 - x_{v-1} x_{v+1}, \tag{3.7d}$$

where r_v and x_v have similar meanings as before, our invariant (3.6) would be dependent on α. The corresponding combinatorial identity would be deprived of its beautiful structure. But, of course, in such a way we can obtain infinitely many combinatorial identities (3.7d) for any cubic irrational in $Q(w)$. Of course, every time $1, \alpha, \alpha^2$ and $1, \alpha^{-1}, \alpha^{-2}$ are to be taken as bases for $Q(w)$.

Now, returning to the powers e^v and e^{-v} in the general cubic cases, the reader will understand that, in principle, there is no structural difference if, in the system of linear equations (3.3a), we take x_v, y_v, z_v as indeterminates and r_v, s_v, t_v as coefficients. Carrying out the same calculations, we would then arrive at a formula, completely analogous to (3.6), viz.,

$$\underline{\underline{x_v = r_v^2 - r_{v-1}r_{v+1}}}; \quad (v = 1, 2, \ldots). \tag{3.8}$$

We shall verify (3.8) for a few values of v. We calculate from (3.1), viz.,

$$x_{v+3} = x_v + a_2 x_{v+1} + a_1 x_{v+2}; \; x_0 = 1, \; x_1 = x_2 = 0;$$
$$r_{v+3} = r_v - a_1 r_{v+1} - a_2 r_{v+2}; \; r_0 = 1, \; r_1 = r_2 = 0.$$
$$x_4 = a_1; \; x_5 = a_2 + a_1; \; x_6 = 1 + 2a_1a_2 + a_1^3.$$
$$r_4 = -a_2; \; r_5 = -a_1^2 + a_1; \; r_6 = 1 + 2a_1a_2 - a_2^3.$$

$$\begin{cases} x_4 = r_4^2 - r_3 r_5, \\ a_1 = a_2^2 - 1(-a_1 + a_2^2) = a_1 \end{cases}$$

$$x_5 = r_5^2 - r_4 r_6,$$

$$\begin{cases} a_2 + a_1^2 = (-a_1 + a_2^2)^2 - (-a_2)(1 + 2a_1a_2 - a_2^3) \\ \qquad = a_1^2 - 2a_1a_2^2 + a_2^4 + a_2 + 2a_1a_2^2 - a_2^4 = a_2 + a_1^2. \end{cases}$$

It exposes the complicated structure of formulas (3.6) and (3.8), if we write out in full these combinatorial identities and substitute the corresponding values for x_v and r_v. We obtain from (1.6) and (1.9b)

$$r_{m+3} = x_{m+3}^2 - x_{m+2}x_{m+4}, \quad (m = 0, 1, \ldots)$$

$$\begin{cases} \displaystyle\sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} (-1)^{m-i-j} \binom{m-i}{i-j}\binom{i-j}{j} a_1^{i-2j} a_2^{m-2i+j} \\ = \left[\displaystyle\sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-i}{i-j}\binom{i-j}{j} a_1^{m-2i+j} a_2^{i-2j}\right]^2 \\ - \left[\displaystyle\sum_{i=0}^{\left[\frac{2(m-1)}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-1-i}{i-j}\binom{i-j}{j} a_1^{m-1-2i+j} a_2^{i-2j}\right] \\ \times \left[\displaystyle\sum_{i=0}^{\left[\frac{2(m+1)}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m+1-i}{i-j}\binom{i-j}{j} a_1^{m+1-2i+j} a_2^{i-2j}\right]; \; a_1a_2 \neq 0. \end{cases} \tag{3.9}$$

(3.9) illustrates the complicity of these combinatorial identities, and it would be a challenging problem to prove it by "elementary" means. In the same way, we obtain

$$x_{m+3} = r_{m+3}^2 - r_{m+2}r_{m+4}, \quad (m = 0, 1, \ldots)$$

$$\left\{\begin{aligned} &\sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} \binom{m-i}{i-j}\binom{i-j}{j} a_2^{m-2i+j} a_1^{i-2j} \\ &= \left[\sum_{i=0}^{\left[\frac{2m}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} (-1)^{m-i-j}\binom{m-i}{i-j}\binom{i-j}{j} a_1^{i-2j} a_2^{m-2i+j}\right]^2 \\ &- \left[\sum_{i=0}^{\left[\frac{2(m-1)}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} (-1)^{m-i-j-1}\binom{m-i-1}{i-j}\binom{i-j}{j} a_2^{i-2j} a_1^{m-1-2i+j}\right] \\ &\times \left[\sum_{i=0}^{\left[\frac{2(m+1)}{3}\right]} \sum_{j=0}^{\left[\frac{i}{2}\right]} (-1)^{m-i-j+1}\binom{m-i+1}{i-j}\binom{i-j}{j} a_2^{i-2j} a_1^{m+1-2i+j}\right]; \ a_1 a_2 \neq 0. \end{aligned}\right. \tag{3.10}$$

Now let

$$\left\{\begin{aligned} &e^3 = 1 + a_2 e, \ a_2 \neq 0, \\ &x_{2m+3} = \sum^{\left[\frac{m}{3}\right]} \binom{m-i}{2i} a_2^{m-3i}, \\ &x_{2m+4} = \sum^{\left[\frac{m-1}{3}\right]} \binom{m-i}{2i+1} a_2^{m-3i-1}, \\ &r_{m+3} = \sum^{\left[\frac{m}{3}\right]} \binom{m-2j}{j} (-1)^{m-j} a_2^{m-3j}, \ m = 0, 1, \ldots. \end{aligned}\right. \tag{3.11}$$

We have

$$r_{2v+3} = x_{2v+3}^2 - x_{2v+2}x_{2v+4}, \tag{3.12}$$

and substituting in (3.12) the values of (3.11), we obtain

$$\begin{cases} \displaystyle\sum_{j=0}^{\left[\frac{2m}{3}\right]} \binom{2m-2j}{j}(-1)^j a_2^{2m-3j} \\ \displaystyle = \left[\sum_{i=0}^{\left[\frac{m}{3}\right]} \binom{m-i}{2i} a_2^{m-3i}\right]^2 \\ \displaystyle - \sum_{i=0}^{\left[\frac{m-2}{3}\right]} \binom{m-1-i}{2i+1} a_2^{m-2-3i} \sum_{i=0}^{\left[\frac{m-1}{3}\right]} \binom{m-i}{2i+1} a_2^{m-3i-1}, \quad (m = 2, 3, \ldots). \end{cases} \tag{3.13}$$

Special cases of (3.11) were investigated by the author in two previous papers [1] and [2], and by L. Carlitz [3] and [4].

The case $a_1 \neq 0$, $a_2 = 0$ is treated analogously.

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