ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, PA 17745

Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS *to* Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvanis 17745. *This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.*

H-290 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Show that:

(a) $F_kF^2_{k+6r+3} - F^3_{k+4r+2} = (-1)^{k+1}F^2_{2r+1}(F_{k+8r+4} - 2F_{k+4r+2})$;

(b) $F_kF^2_{k+6r} - F^3_{k+4r} = (-1)^{k+1}F^2_{2r}(F_{k+8r} + 2F_{k+4r})$.

H-291 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX*

Prove that there are infinitely many squares which are differences of consecutive cubes.

H-292 *Proposed by F. S. Cater and J. Daily, Portland State University, Portland, OR.*

Find all real numbers $r \in (0,1)$ for which there exists a one-to-one function f_r mapping $(0,1)$ onto $(0,1)$ such that

(1) f_r and f_r^{-1} are infinitely many times differentiable on $(0,1)$, and
(2) the sequence of functions f_r, $f_r \circ f_r$, $f_r \circ f_r \circ f_r$, $f_r \circ f_r \circ f_r \circ f_r$, ... converges pointwise to r on $(0,1)$.

H-293 *Proposed by Leonard Carlitz, Duke University, Durham, NC.*

It is known that the Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ defined by

$$\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!} = e^{2xz - z^2}$$

satisfy the relation

$$\sum_{n=0}^{\infty} H_{n+k}(x)\frac{z^n}{n!} = e^{2xz - z^2}H_k(x - z) \quad (k = 0,\ 1,\ 2,\ \ldots).$$

Show that conversely if a set of polynomials $\{f_n(x)\}_{n=0}^{\infty}$ satisfy

$$(1) \qquad \sum_{n=0}^{\infty} f_{n+k}(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} f_n(x)\frac{z^n}{n!}f_k(x - z) \quad (k = 0,\ 1,\ 2,\ \ldots),$$

where $f_0(x) = 1$, $f_1(x) = 2x$, then

$$f_n(x) = H_n(x) \quad (n = 0,\ 1,\ 2,\ \ldots).$$

H-294 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Evaluate

$$\Delta = \begin{vmatrix} F_{2r+1} & F_{6r+3} & F_{10r+5} & F_{14r+7} & F_{18r+9} \\ F_{4r+2} & F_{12r+6} & F_{20r+10} & F_{28r+14} & F_{36r+18} \\ F_{6r+3} & F_{18r+9} & F_{36r+15} & F_{42r+21} & F_{54r+27} \\ F_{8r+4} & -F_{24r+12} & F_{40r+20} & F_{56r+28} & F_{72r+36} \\ F_{10r+5} & F_{20r+15} & F_{50r+25} & F_{70r+36} & F_{50r+45} \end{vmatrix}$$

SOLUTIONS

SYMMETRIC SUM

H-272 (Corrected) *Proposed by Leonard Carlitz, Duke University, Durham, NC.*

Show that

$$\sum_{j=0}^{m} \binom{r}{j}\binom{p}{m-j}\binom{q}{m-j}\binom{p+q-m+j}{j} \Big/ \binom{m}{j} \equiv C_m(p,\, q,\, r)$$

is symmetric in p, q, r.

Solution by Paul Bruckman, Concord, CA.

Define

$$(1) \qquad C_m(p,\, q,\, r) = \sum_{j=0}^{m} \binom{r}{j}\binom{p}{m-j}\binom{q}{m-j}\binom{p+q-m+j}{j} \Big/ \binom{m}{j}.$$

Clearly, $C_m(p,\, q,\, r) = C_m(q,\, p,\, r)$. A moment's reflection reveals that it therefore suffices to show that $C_m(p,\, q,\, r) = C_m(q,\, r,\, p)$. Replacing j by $m - j$ in (1) and applying Vandermonde's convolution theorem on the term involving p and q yields:

$$\begin{aligned} C_m(p,\, q,\, r) &= \sum_{j=0}^{m} \binom{r}{m-j}\binom{p}{j}\binom{q}{j} \Big/ \binom{m}{j} \sum_{k=0}^{m-j} \binom{p-j}{m-j-k}\binom{q}{k} \\ &= \sum_{j=0}^{m} \sum_{k=0}^{m-j} \binom{r}{m-j}\binom{q}{j}\binom{q}{k}\binom{p}{m-k}\binom{m-k}{j} \Big/ \binom{m}{j}. \end{aligned}$$

Replacing k by $m - k$ in the last expression yields:

$$\begin{aligned} C_m(p,\, q,\, r) &= \sum_{j=0}^{m} \sum_{k=j}^{m} \binom{r}{m-j}\binom{q}{j}\binom{q}{m-k}\binom{p}{k}\binom{k}{j} \Big/ \binom{m}{j} \\ &= \sum \binom{p}{k}\binom{q}{m-k} \sum \binom{r}{m-j}\binom{q}{j}\binom{k}{j} \Big/ \binom{m}{j}. \end{aligned}$$

However, it is easy to verify that

$$(2) \qquad \binom{r}{m-j} \Big/ \binom{m}{j} = \binom{r}{m} \Big/ \binom{r-m+j}{j} = \binom{r}{m}(-1)^j \Big/ \binom{m-r-1}{j}.$$

Therefore,

$$(3) \qquad C_m(p,\, q,\, r) = \binom{r}{m} \sum_{k=0}^{m} \binom{p}{k}\binom{q}{m-k} \sum_{j=0}^{k} (-1)^j \binom{k}{j}\binom{q}{j} \Big/ \binom{m-r-1}{j}.$$

Now, formula (7.1) in *Combinatorial Identities* (H. W. Gould, Morgantown, 1972), is as follows:

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{z}{k} \Big/ \binom{y-z}{n} \Big/ \binom{y}{n}. \tag{4}$$

Letting $k = j$, $n = k$, $z = q$, $y = m - r - 1$ in (4), we therefore simplify (3) as follows:

$$C_m(p,\ q,\ r) = \binom{r}{m}\sum_{k=0}^{m}\binom{p}{k}\binom{q}{m-k}\binom{m-r-q-1}{k}\Big/\binom{m-r-1}{k}$$

$$= \binom{r}{m}\sum_{k=0}^{m}\binom{p}{k}\binom{q}{m-k}\binom{q+r-m+k}{k}\Big/\binom{r-m+k}{k};$$

now using (2) once again and replacing k by j yields:

$$C_m(p,\ q,\ r) = \sum_{j=0}^{m}\binom{p}{j}\binom{q}{m-j}\binom{r}{m-j}\binom{q+r-m+j}{j}\Big/\binom{m}{j}$$

$$= C_m(q,\ r,\ p). \quad \text{Q.E.D.}$$

Also solved by the proposer.

A RAY OF LUCAS

H-273 *Proposed by W. G. Brady, Slippery Rock State College, Slippery Rock, PA.*

Consider, after Hoggatt and H-257, the array D, indicated below, in which L_{2n+1} ($n = 0, 1, 2, \ldots$) is written in staggered columns:

1				
4	1			
11	4	1		
29	11	4	1	
76	29	11	4	1

i. Show that the row sums are $L_{2n+2} - 2$;

ii. Show that the rising diagonal sums are $F_{2n+3} - 1$ where L_{2n+1} is the largest element in the sum.

iii. Show that if the columns are multiplied by 1, 2, 3, ... sequentially to the right then the row sums are $L_{2n+3} - (2n + 3)$.

Solution by A. G. Shannon, The N.S.W. Institute of Technology, Australia.

In effect we are asked to prove:

i. $\displaystyle\sum_{j=0}^{n} L_{2n-2j+1} = L_{2n+2} - 2;$

ii. $\displaystyle\sum_{j=0}^{[n/2]} L_{2n-4j+1} = F_{2n+3} - 1;$

iii. $\displaystyle\sum_{j=0}^{n} (j+1)L_{2n-2j+1} = L_{2n+3} - (2n+3).$

(i) $$\sum_{j=0}^{n} L_{2n-2j+1} = \sum_{j=0}^{n}\left(L_{2n-2(j-1)} - L_{2n-2j}\right) = \sum_{j=0}^{n} L_{2n-2(j-1)} - \sum_{j=1}^{n+1} L_{2n-2(j-1)}$$

$$= L_{2n+2} - L_0, \text{ as required.}$$

(ii) $$\sum_{j=0}^{[n/2]} L_{2n-4j+1} = \sum_{j=0}^{[n/2]} \left(F_{2n-4j+3} - F_{2n-4j-1}\right) = \sum_{j=0}^{[n/2]} F_{2n-4j+3} - \sum_{j=1}^{[n/2]+1} F_{2n-4j+3}$$

$$= F_{2n+3} - F_1\sigma(2, n) - F_{-1}\sigma(2, n+1) = F_{2n+3} - 1$$

in which

$$\sigma(n, m) = \begin{cases} 1 & \text{if } n \mid m, \\ 0 & \text{if } n \nmid m. \end{cases}$$

(iii) $$\sum_{j=0}^{n} (j+1)L_{2n-2j+1} = \sum_{i=0}^{n}\sum_{j=1}^{n} L_{2n-2j+1} = \sum_{i=0}^{n}\left(L_{2n-2i+2} - 2\right)$$

[from (i)]

$$= \sum_{i=0}^{n}\left(L_{2n-2i+3} - L_{2n-2i+1} - 2\right)$$

$$= \sum_{i=0}^{n+1} L_{2(n+1)-2i+1} - L_1 - \left(L_{2n+2} - 2\right) - 2(n+1)$$

$$= L_{2n+4} - 2 - 1 - L_{2n+2} + 2 - 2(n+1)$$

$$= L_{2n+3} - (2n+3), \text{ as required.}$$

Also solved by P. Bruckman, G. Wulczyn, H. Freitag, B. Prielipp, Dinh Thê'Hūng, and the proposer.

Late Acknowledgments: F. T. Howard solved H-268 and M. Klamkin solved H-270.

A CORRECTED OLDIE

H-225 *Proposed by G. A. R. Guillotte, Quebec, Canada.*

Let p denote an odd prime and $x^p + y^p = z^p$ for positive integers, x, y, and z. Show that

(A) $p < x/(z - x) + y/(z - y)$, and

(B) $z/2(z - x) < p < y/(z - y)$.

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