PRIMES, POWERS, AND PARTITIONS

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Elementary arguments are employed in this paper to give a characterization of the set of primes and to extend this set to a larger one whose elements are defined by a single property: we show that a positive integer is either a prime or a power of 2 if and only if such an integer cannot be expressed as a sum of at least three consecutive positive integers. This fact provides an easy sieve to isolate the primes (and if one prefers, the immediately recognizable powers of 2) less than or equal to any preassigned positive integer. We also describe the possible ways in which a given composite number may be expressed as a sum of at least three consecutive positive integers; such representations for the sake of brevity shall be termed σ-partitions of the given integers. Furthermore, "number" shall mean "positive integer" and the set of all these numbers will, as usual, be denoted by $\mathbb{N}$.

Lemma 1: An odd number m admits a σ-partition if and only if m is a composite number.

Proof ($\Rightarrow$): Let $m = n + (n + 1) + \cdots + (n + k)$, $n \in \mathbb{N}$, $k \geq 2$. Then,

$$m = \frac{k+1}{2}(2n + k).$$

If $k + 1$ is even, then $(k + 1)/2$ is an odd number ≥ 3, since $k \geq 3$ in this case; obviously, therefore, $2n + k$ is an odd number ≥ 5. Hence, m is a composite number. If $k + 1$ is odd, then k is even, and since $k \geq 2$, one must have that $2n + k$ is an even number ≥ 4. Since m is an odd number, it follows that $(2n + k)/2$ is an odd number > 2. The fact that $k + 1 \geq 3$ now shows that m is a composite number.

Proof ($\Leftarrow$): Consider an arbitrary factorization $m = kg$, $(3 \leq k \leq g)$. Then,

$$m = kg = \frac{k}{2}(2g) = \frac{k}{2}\left[2\left(g - \frac{k-1}{2}\right) + k - 1\right] = \frac{k}{2}(2a + k - 1)$$

where

$$a = g - \frac{k-1}{2} \in \mathbb{N}.$$

Hence, we have

$$m = \sum_{r=1}^{k} (a + r - 1),$$

which is a σ-partition, since $a \in \mathbb{N}$ and $k \geq 3$.

Corollary 1: An odd number is prime if and only if it admits no σ-partition.

Lemma 2: An even number m admits a σ-partition if and only if m is not a power of 2. (Cf. [1], p. 17.)

Proof ($\Rightarrow$): Let $m = n + (n + 1) + \cdots + (n + k)$; $n \in \mathbb{N}$, $k \geq 2$. Then,

$$m = \frac{k+1}{2}(2n + k).$$

If $m = 2^s$, then, since $k \geq 2$, we must have that $k + 1 = 2^t$, $t \geq 2$, and $2n + k = 2^u$, $u \geq 2$. This is a contradiction, since $k + 1 = 2^t$ would imply that k is odd, so that $2n + k$ would also be odd. Hence, we have that m is not a power of 2.

Proof ($\Leftarrow$): Suppose that m is not a power of 2. Set $m = 2\ v$, $n \geq 1$, v an odd number ≥ 3.

Case (i): $v < 2^n$. Write $k = v$ and $g = 2^n$. Then,

$$m = kg = \frac{k}{2}(2g) = \frac{k}{2}\left[2\left(g - \frac{k-1}{2}\right) + k - 1\right] = \frac{k}{2}(2a + k - 1),$$

where

$$a = g - \frac{k-1}{2} \in \mathbb{N}.$$

Thus we have

$$m = \sum_{r=1}^{k} (a + r - 1)$$

which is clearly a σ-partition.

Case (ii): $v > 2^n$. Write $k = 2^n$ and $g = v$. Now,

$$g \leq 2k - 1 \Rightarrow k \geq \frac{g+1}{2} \Rightarrow k > \frac{g-1}{2} \Rightarrow k - \frac{g-1}{2} \in \mathbb{N},$$

and we have

$$m = gk = \frac{g}{2}(2k) = \frac{g}{2}\left[2\left(k - \frac{g-1}{2}\right) + g - 1\right] = \frac{g}{2}(2a + g - 1)$$

where

$$a = k - \frac{g-1}{2}.$$

Hence,

$$m = \sum_{r=1}^{g} (a + r - 1),$$

a σ-partition. On the other hand,

$$g > 2k - 1 \Rightarrow \frac{g+1}{2} > k \Rightarrow \frac{g+1}{2} - k \in \mathbb{N},$$

and now

$$m = \frac{2k}{2}g = \frac{2k}{2}\left[2\left(\frac{g+1}{2} - k\right) + 2k - 1\right] = \sum_{r=1}^{2k} (a + r - 1)$$

where

$$a = \frac{g+1}{2} - k \in \mathbb{N}, \text{ and clearly } 2k \geq 4.$$

This completes the proof.

Corollary 2: An even number is a power of 2 if and only if it admits no σ-partition.

We now have a natural extension of the sequence of primes, defined by a single property, in the following direct consequence of our two corollaries.

Theorem 1: A number m is either a prime or a power of 2 if and only if m admits no σ-partition.

This theorem provides an easy sieve to isolate the set of primes (and immediately recognizable powers of 2) less than or equal to any preassigned number x of moderate size: one simply writes down the segment $1, 2, 3, \ldots, x$ and crosses out the σ-partitions less than or equal to x, starting with those with leading term $a = 1$, then those with $a = 2$, etc. The least upper bound for the set of these a's is the number $\left[\frac{x-3}{3}\right]$, for clearly $a \lesseqgtr \frac{x-3}{3}$ if and only if $3a + 3 \lesseqgtr x$, which is equivalent to $a + (a + 1) + (a + 2) \lesseqgtr x$.

A simple example of "sieving" out the primes, for instance with $x = 15$, shows that a given composite number may admit more than one σ-partition. Our next problem is to give an account of the different possible σ-partitions of a given composite number m. First, we deal with the case where m is an odd number. Consider an arbitrary factorization $m = kg$ $(3 \le k \le g)$. Corresponding with this factorization, one always has the σ-partition

$$(1) \qquad \sum_{r=1}^{k}\left[\left(g - \frac{k-1}{2}\right) + r - 1\right] \text{ of } m.$$

If $g < 2k + 1$, then $k - \frac{g-1}{2} \in \mathbb{N}$, and once again direct computation shows that

$$(2) \qquad \sum_{r=1}^{g}\left[\left(k - \frac{g-1}{2}\right) + r - 1\right]$$

is a σ-partition of m. Clearly, (2) coincides with the fixed partition (1) if and only if $k = g$, i.e., the case where m is a square and is factored as such.

If $g \ge 2k + 1$, then clearly $\frac{g+1}{2} - k \in \mathbb{N}$, and we obtain the σ-partition

$$(3) \qquad \sum_{r=1}^{2k}\left[\left(\frac{g+1}{2} - k\right) + r - 1\right] \text{ of } m.$$

The partitions (1) and (3), having different lengths, can never coincide.

Conversely, the indicated possible σ-partitions corresponding to the particular type of factorization of m are the only possible ones m can have. For, consider an arbitrary σ-partition

$$m = \sum_{r=1}^{n}(a + r - 1) = \frac{n}{2}(2a + n - 1),\ n \ge 3.$$

If n is even, then $n/2$ is an odd divisor of m and so is $2a + n - 1$. Clearly, $2a + n - 1 > n > n/2$ so that we may write $k = n/2$ and $g = 2a + n - 1$, $k < g$. In this notation, we have that $g = 2a + 2k - 1 \ge 2k + 1$, since $2a \ge 2$, and $a = (g + 1)/2 - k$. Hence, the given partition is of the form (3). If n is an odd number, we have that $2a + n - 1$ is even and that $w = (2a + n - 1)/2$ is an odd divisor of m. If $n \le w$, we put $k = n$ and $g = w$. Then, $2g = 2a + k - 1$, so that $a = g - (k - 1)/2$, and the given partition has the form (1). If $n > w$, we write $k = w$ and $g = n$. Then, $2k = 2a + g - 1$, so that $g = 2k + 1 - 2a < 2k + 1$, since $a \ge 1$, and $a = k - (g - 1)/2$. This shows that the given partition has the form (2).

Summarizing these observations, we obtain the following characterization of the σ-partitions of a given composite odd number.

Theorem 2: The σ-partitions of a composite odd number m are precisely those determined by the factorizations of the form $m = kg$ $(3 \le k \le g)$, namely

$$(1) \qquad \sum_{r=1}^{k}\left[\left(g - \frac{k-1}{2}\right) + r - 1\right];$$

and exactly one of

$$(2) \qquad \sum_{=1}^{g}\left[\left(k - \frac{g-1}{2}\right) + r - 1\right] \quad (\text{if } g < 2k + 1)$$

and

$$(\ )\qquad \sum_{r=1}^{2k}\left[\left(\frac{g+1}{2}-k\right)+r-1\right]\quad (\text{if } g \geq 2k+1).$$

If $g < 2k + 1$, the two valid partitions (*1*) are (*2*) are different, except in the case where $g = k$, and if $g \geq 2k + 1$, the valid partitions (*1*) and (*3*) are always different.

Now consider any two different factorizations (if they exist) of m into two factors: $m = kg$ $(3 \leq k \leq g)$ and $m = k'g'$ $(3 \leq k' \leq g')$. Then, comparing the lengths of the resulting σ-partitions, one sees that every possible partition corresponding with $m = kg$ differs from every possible one corresponding with $m = k'g'$. Hence, if m admits t different factorizations of the form $m = kg$ $(3 \leq k \leq g)$, then m admits $2t$ different σ-partitions, except in the case where m is a square in which case the number is $2t - 1$.

Finally, we consider the nature and number of σ-partitions of even numbers other than powers of 2.

Theorem 3: Let m be an even number other than a power of 2. Then, there exists at least one factorization of the form $m = kg$ $(k < g)$, where one of the factors is an even number and the other an odd number ≥ 3. For each such factorization, exactly one of the following three conditions holds:

(*1*) k is even, g is odd and $g < 2k + 1$;
(*2*) k is even, g is odd and $g \geq 2k + 1$;
(*3*) k is odd, g is even;

and only that sum in the list

$$(1')\qquad \sum_{r=1}^{g}\left[\left(k-\frac{g-1}{2}\right)+r-1\right],$$

$$(2')\qquad \sum_{r=1}^{2k}\left[\left(\frac{g+1}{2}-k\right)+r-1\right],$$

$$(3')\qquad \sum_{r=1}^{k}\left[\left(g-\frac{k-1}{2}\right)+r-1\right],$$

which corresponds with the valid condition is a σ-partition of m. Finally, these are the only possible types of σ-partitions of m.

Proof: First we observe that m may be written in the form $m = 2^n u$, where $n \geq 1$ and u is an odd number ≥ 3. Now, consider an arbitrary factorization, $m = kg$ $(k < g)$, into an even and an odd factor. If k is even, then the possibility (*3*) is ruled out and the ordering axiom ensures that exactly one of (*1*) and/or (*2*) holds. If k is odd, then the first two possibilities are excluded and (*3*) obviously holds.

Concerning the next part of Theorem 3, we note that each of the indicated sums is of the form

$$\sum_{r=1}^{n}(a+r-1),$$

and that $n \geq 3$ and $a \in \mathbb{N}$, providing that the condition to which the particular sum corresponds holds. Moreover, in each of these cases the indicated summation results in the product kg. On the other hand, the validity of any given condition (i) clearly results in $a \notin \mathbb{N}$ in the sums (j'), $i \neq j$. This concludes this part of the proof.

Finally, we consider an arbitrary σ-partition

$$m = \sum_{r=1}^{n} (a + r - 1) = \frac{n}{2}(2a + n - 1).$$

If n is even, then $2a + n - 1$ is an odd divisor of m, and since m is even, one must have that n contains a factor 4. Since $2a + n - 1 > n > n/2$, we may write $k = n/2$ and $g = 2a + n - 1$. Then $m = kg$, $k < g$, k is even and g is odd. Moreover, $g > 2k$, so $g \geq 2k + 1$. Finally, from $g = 2a + 2k - 1$, we obtain

$$a = \frac{g + 1}{2} - k.$$

Hence, the given partition has the form (*2′*). If n is odd, then n is a divisor of m and $2a + n - 1$ is an even number. Since m is even, we must have that $(2a + n - 1)/2$ is an even divisor of m and we may write $(2a + n = 1)/2 = 2w$. Considering first the case where $n > 2w$ and using the notation $g = n$ and $k = 2w$, one easily checks that k and g satisfy the requirements of condition (*1*) and that the given partition has the form (*1′*). A similar straightforward analysis of the case $n < 2w$ shows that $k = n$ and $g = 2w$ satisfy condition (*3*) and that the given partition has the form (*3′*). This completes the proof.

In conclusion, we want to determine the number of different σ-partitions of an even number m other than a power of 2. We once again consider two different factorizations as specified in the theorem (if they exist):

(α) $m = kg$,
(β) $m = k'g'$.

Let condition (i) in the theorem be satisfied in (α), and let condition (j) be satisfied in (β). We consider two possibilities:

$i = j$: Here k and k' are both even or both odd. Since $k \neq k'$ and $g \neq g'$ we must have that the σ-partition (i') relative to (α) is different from the σ-partition (j') = (i') relative to (β).

$i \neq j$: Suppose that k and k' are both even. Then g and g' are both odd and of the resulting σ-partitions one is of the form (*1′*) and the other of the form (*2′*). Noting that one of the lengths here is odd and the other one even, we conclude that the two σ-partitions are different. Suppose now that one of the factors k and k' is even and the other one odd. Without loss of generality we may assume that k is even. Then $j = 3$ and the factorization (β) yields the σ-partition (*3′*) of odd length. If $i = 1$, then the equality of (*1′*) relative to (α) and (*3′*) relative to (β) would imply that $g = k'$, so that $k = g'$. This, however, would imply that $k > g$, a contradiction. Hence, we have that the two resulting σ-partitions are different in this case as well. If $i = 2$, then the partition (*2′*) relative to (α) has even length, while (*3′*) relative to (β) has odd length, so they do not coincide. Therefore, we may conclude that if m admits t factorizations $m = kg$ where one of the factors is an even number and the other one an odd number ≥ 3, then m admits t different σ-partitions.

REFERENCE

1. I. Niven & H. S. Zuckerman, *An Introduction to the Theory of Numbers*, 3rd ed. (New York: John Wiley & Sons, 1972).

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