The Values of λ_k and ε_k, $1 \leq k \leq 10$

k	λ_k	ε_k
1	0	0
2	0	0
3	0	0
4	2	0.066667
5	6	0.041667
6	8	0.008929
7	14	0.002401
8	17	0.000375
9	26	0.000064
10	39	0.000009

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EVALUATION OF SUMS OF CONVOLVED POWERS USING STIRLING AND EULERIAN NUMBERS

H. W. GOULD

West Virginia University, Morgantown, W. Va. 26506

ABSTRACT

It is shown here how the method of generating functions leads quickly to compact formulas for sums of the type

$$S(i,j;n) = \sum_{0 \leq k \leq n} k^i (n - k)^j$$

using Stirling numbers of the second kind and also using Eulerian numbers. The formulas are, for the most part, much simpler than corresponding results using Bernoulli numbers.

1. INTRODUCTION

Neuman and Schonbach [9] have obtained a formula for the series of convolved powers

$$S(i,j;n) = \sum_{k=0}^{n} k^i (n - k)^j \tag{1.1}$$

using Bernoulli numbers. Although the formula expresses $S(i,j;n)$ as a polynomial of degree $i + j + 1$ in n, and this mode of expression is useful, still the formula is rather clumsy and hard to recall. Below we shall show how the method of generating functions can be used to obtain elegant closed forms for (1.1) very quickly. The first of these uses the Stirling numbers of the second kind, and the second uses the Eulerian numbers. Both results give (1.1) as series of binomial coefficients in n, rather than directly as polynomials expressed explicitly in powers of n. For many purposes of computation and number theoretic study, such expressions are desirable. The significant results below are formulas (3.6), (3.8), (5.3), and (7.3).

Glaisher [4] and [5] was the first to sum (1.1) using Bernoulli numbers. Carlitz [3] has shown some extensions of [9] and connections with Eulerian numbers. Our results overlap some of those of Carlitz, but were obtained in August 1974 before [3] was written.

2. A GENERATING FUNCTION

$$G(t;i,j) = \sum_{n=0}^{\infty} t^n S(i,j;n). \tag{2.1}$$

Then

$$G(t;i,j) = \sum_{k=0}^{\infty} k^i \sum_{n=k}^{\infty} t^n (n-k)^j = \sum_{k=0}^{\infty} k^i \sum_{n=0}^{\infty} t^{n+k} n^j,$$

so that we have at once the elegant generating function

$$G(t;i,j) = \sum_{k=0}^{\infty} k^i t^k \cdot \sum_{n=0}^{\infty} n^j t^n. \tag{2.2}$$

The generalized power series

$$\sum_k k^p t^k$$

may be summed in a variety of ways. We shall use the methods of (i) Stirling numbers of the second kind and (ii) Eulerian numbers. Our (2.2) is (3.4) in Carlitz [3].

3. METHOD OF STIRLING NUMBERS OF THE SECOND KIND

It is an old fact that

$$(tD)^p f(t) = \sum_{k=0}^{p} S(p,k) t^k D^k f(t), \tag{3.1}$$

where $D = d/dt$ and $S(p,k)$ is a Stirling number of the second kind. Explicitly,

$$k!S(p,k) = \Delta^k 0^p = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} k^p. \tag{3.2}$$

The formula dates back more than 150 years, but, for a recent example, see Riordan [10, p. 45, ex. 18]. Riordan gives a full account of the properties of Stirling numbers of both first and second kinds. Other historical remarks and variant notations are discussed in [6]. Applying the formula is easy because $(tD)^p t^k = k^p t^k$, whence we have

$$(3.3)\qquad \sum_{k=0}^{\infty} k^p t^k = \sum_{k=0}^{p} k!S(p,k)\frac{t^k}{(1-t)^{k+1}}.$$

This, too, is a very old formula. It converges for $|t| < 1$, but we treat it as a formal power series. Carlitz [2] gives a good discussion of formal power series techniques.

Using (3.3) in (2.2), we find

$$(3.4)\qquad G(t;i,j) = \sum_{r=0}^{i+j} \frac{t^r}{(1-t)^{r+2}}\sum_{k=0}^{r} k!(r-k)!S(i,k)S(j,r-k).$$

Throughout the rest of the paper, we shall write, for brevity,

$$(3.5)\qquad \mathcal{S}_r(i,j) = \sum_{k=0}^{r} k!(r-k)!S(i,k)S(j,r-k).$$

Applying the binomial theorem, we find next

$$\begin{aligned} G(t;i,j) &= \sum_{r=0}^{i+j}\sum_{n=0}^{\infty}\binom{n+r+1}{r+1}t^{n+r}\mathcal{S}_r(i,j) \\ &= \sum_{r=0}^{i+j}\sum_{n=r}^{\infty}\binom{n+1}{r+1}t^n\mathcal{S}_r(i,j) \\ &= \sum_{n=0}^{\infty} t^n\sum_{r=0}^{n}\binom{n+1}{r+1}\mathcal{S}_r(i,j). \end{aligned}$$

In the next-to-last step here, the upper limit $r = i + j$ might as well have been $r = \infty$ because of zero terms involved, since $S(p,k) = 0$ when $k > p$. This makes manipulation easier. Equating coefficients of t^n and dropping some zero terms, we find finally then our desired formula

$$(3.6)\qquad S(i,j;n) = \sum_{r=0}^{i+j}\binom{n+1}{r+1}\mathcal{S}_r(i,j).$$

This simple expression may be compared with the bulky form of expression given in [9] using Bernoulli numbers.

Having found our desired formula, we can next offer a much quicker proof. Recall [10, p. 33] that

$$(3.7)\qquad x^n = \sum_{r=0}^{n}\binom{x}{r}r!S(n,r).$$

This gives at once

$$k^i(n-k)^j = \sum_{r=0}^{i} r!S(i,r)\sum_{s=0}^{j} s!S(j,s)\binom{k}{r}\binom{n-k}{s},$$

whence, using formula (3.3) in [8], a modified Vandermonde addition formula, we get on summing from $k = 0$ to $k = n$,

$$(3.8)\qquad S(i,j;n) = \sum_{r=0}^{i} r!S(i,r)\sum_{s=0}^{j} s!S(j,s)\binom{n+1}{r+s+1}.$$

By simply putting $s - r$ for s and interchanging the summation order, we see that this is nothing other than our former result (3.6).

4. EXAMPLES OF THE STIRLING NUMBER METHOD

For the sake of completeness, we recall [10, p. 48] some of the values of $S(n,k)$:

	0	1	2	3	4	5	6	7 ⋯ k
0	1							
1		1						
2		1	1					
3		1	3	1				
4		1	7	6	1			
5		1	15	25	10	1		
6		1	31	90	65	15	1	
7		1	63	301	350	140	21	1
⋮								
n								

Here, $S(n,k) = 0$ when $k > n$ and $S(n,0) = 0$ for $n \geq 1$.

For $j = 0$, formula (3.6) becomes the well known

$$S(i,0;n) = \sum_{r=0}^{i} \binom{n+1}{r+1} r! S(i,r), \quad n \geq 0, \quad i \geq 0. \tag{4.1}$$

Incidentally, in some places in the vast literature $r!S(i,r)$ has been called a Stirling number, and both arrays turn up very often in odd places with new notations. There are at least 50 notations for Stirling numbers. Here are a few examples of (4.1):

$$S(1,0;n) = \binom{n+1}{2},$$

$$S(2,0;n) = \binom{n+1}{2} + 2\binom{n+1}{3},$$

$$S(3,0;n) = \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4},$$

$$S(4,0;n) = \binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5}.$$

For $j = 1$ we shall obtain substantially the same coefficients, the difference being that the lower indices are each increased by 1. Thus:

$$S(2,1;n) = \binom{n+1}{3} + 2\binom{n+1}{4} = \frac{n^4 - n^2}{12},$$

$$S(3,1;n) = \binom{n+1}{3} + 6\binom{n+1}{4} + 6\binom{n+1}{5} = \frac{3n^5 - 5n^3 + 2n}{60},$$

$$S(4,1;n) = \binom{n+1}{3} + 14\binom{n+1}{4} + 36\binom{n+1}{5} + 24\binom{n+1}{6}$$
$$= \frac{2n^6 - 5n^4 + 3n^2}{60},$$

where we have indicated, for comparison, the values obtained in [9].

For $j = 1$, the following is a brief table of the coefficients in the array:

$i = 2$:	1	2				
$i = 3$:	1	6	6			
$i = 4$:	1	14	36	24		
$i = 5$:	1	30	150	240	120	
$i = 6$:	1	62	450	1560	1800	720

For $j = 3$, we find the following formulas:

$$S(0,3;n) = \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4},$$

$$S(1,3;n) = \binom{n+1}{3} + 6\binom{n+1}{4} + 6\binom{n+1}{5},$$

$$S(2,3;n) = \binom{n+1}{3} + 8\binom{n+1}{4} + 18\binom{n+1}{5} + 12\binom{n+1}{6},$$

$$S(3,3;n) = \binom{n+1}{3} + 12\binom{n+1}{4} + 48\binom{n+1}{5} + 72\binom{n+1}{6} + 36\binom{n+1}{7},$$

and so forth.

5. METHOD OF EULERIAN NUMBERS

The Eulerian numbers [1], [10, pp. 39, 215] are given by

$$A_{n,j} = \sum_{k=0}^{j} (-1)^k \binom{n+1}{k}(j-k)^n. \tag{5.1}$$

These must not be confused with Euler numbers appearing in the power series expansion of the secant function. The Eulerian numbers satisfy

$$A_{n,j} = A_{n,n-j+1}, \text{ row symmetry, } n \geq 1,$$

$$A_{n,j} = jA_{n-1,j} + (n - j + 1)A_{n-1,j-1},$$

and

$$\sum_{j=1}^{n} A_{n,j} = n!.$$

Again, for completeness, here is a brief table of $A_{n,j}$:

	0	1	2	3	4	5	6	7	$\cdots$ j
0	1								
1		1							
2		1	1						
3		1	4	1					
4		1	11	11	1				
5		1	26	66	26	1			
6		1	57	302	302	57	1		
7		1	120	1191	2416	1191	120	1	
$\vdots$									
n									

These numbers are frequently rediscovered, for example, recently by Voelker [11] and [12], where no mention is made of the vast literature dealing with these numbers and tracing back to Euler. For our purposes, we need the well-known expansion

$$\sum_{k=0}^{\infty} k^n t^k = (1-t)^{-n-1} \sum_{k=0}^{n} t^k A_{n,k}. \tag{5.2}$$

This expansion is known to be valid for $|t| < 1$, but again we treat all series here as formal power series since we do not use the sums of any infinite series. We never assign t a value, but equate coefficients only.

Applying this to (2.2), we find

$$\begin{aligned} G(t;i,j) &= (1-t)^{-i-j-2} \sum_{r=0}^{i} t^r A_{i,r} \sum_{s=0}^{j} t^s A_{j,s} \\ &= \sum_{k=0}^{\infty} \binom{i+j+k+1}{k} t^k \sum_{r=0}^{i+j} t^r \sum_{s=0}^{r} A_{i,s} A_{j,r-s} \\ &= \sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} \binom{i+j+n-r+1}{n-r} \sum_{s=0}^{r} A_{i,s} A_{j,r-s} \end{aligned}$$

and by comparison of coefficients of t^n we have our desired formula

$$S(i,j;n) = \sum_{r=0}^{i+j} \binom{i+j+n-r+1}{i+j+1} \sum_{s=0}^{r} A_{i,s} A_{j,r-s}. \tag{5.3}$$

Here we have again dropped some of the terms that are zero by noting that $A_{n,j} = 0$ whenever $j > n$. Formula (5.3) is (3.6) in Carlitz [3].

As with our previous Stirling number argument, we could obtain (5.3) by another method. We recall that in fact

$$x^n = \sum_{j=0}^{n} \binom{x+j-1}{n} A_{n,j} \tag{5.4}$$

and form the product $k^i(n-k)^j$ and sum from $k=0$ to $k=n$ to obtain a formula for (5.3) analogous to (3.8). We omit the details.

6. EXAMPLES OF THE EULERIAN NUMBER METHOD

When $j = 0$, formula (5.3) becomes, of course, the familiar relation

$$S(i,0;n) = \sum_{r=1}^{i} \binom{n+r}{i+1} A_{i,r}, \quad n \geq 0, \; i \geq 1. \tag{6.1}$$

To see that this is so, we proceed as follows. By (5.3),

$$\begin{aligned} S(i,0;n) &= \sum_{r=0}^{i} \binom{i+n-r+1}{i+1} \sum_{s=0}^{r} A_{i,s} A_{0,r-s} \\ &= \sum_{r=0}^{i} \binom{i+n-r+1}{i+1} A_{i,r}, \text{ since } A_{0,r-s} = 0 \text{ for } r \neq s, \end{aligned}$$

$$= \sum_{r=1}^{i} \binom{i+n-r+1}{i+1} A_{i,r}, \text{ since } A_{i,0} = 0 \text{ for } i \geq 1,$$

$$= \sum_{r=1}^{i} \binom{n+r}{i+1} A_{i,i-r+1}, \text{ by putting } i - r + 1 \text{ for } r,$$

$$= \sum_{r=1}^{i} \binom{n+r}{i+1} A_{i,r}, \text{ by the symmetry relation.}$$

For $j = 0$, then, we have the following formulas:

$$S(1,0;n) = \binom{n+1}{2},$$

$$S(2,0;n) = \binom{n+1}{3} + \binom{n+2}{3},$$

$$S(3,0;n) = \binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4}$$

$$S(4,0;n) = \binom{n+1}{5} + 11\binom{n+2}{5} + 11\binom{n+3}{5} + \binom{n+4}{5},$$

etc.

For $j = 1$, we find

$$S(2,1;n) = \binom{n+1}{4} + \binom{n+2}{4}$$

$$S(3,1;n) = \binom{n+1}{5} + 4\binom{n+2}{5} + \binom{n+3}{5}$$

$$S(4,1;n) = \binom{n+1}{6} + 11\binom{n+2}{6} + 11\binom{n+3}{6} + \binom{n+4}{6}$$

and so on. These again are a different way of saying what was found in [9].

7. ALTERNATIVE EXPRESSION OF THE STIRLING NUMBER EXPANSION

Formula (3.6) uses the values of $\binom{n+1}{r+1}$. We wish to show now that we can transform this result easily into a formula using just $\binom{n}{r+1}$, i.e., directly as a series of binomial coefficients in n rather than $n + 1$. We will need to recall, see [10], the recurrence relation for Stirling numbers of the second kind

(7.1) $$S(m,k) = kS(m-1,k) + S(m-1,k-1).$$

In this, set $m = j + 1$ and replace k by $r - k$. We get

(7.2) $$S(j+1,r-k) = (r-k)S(j,r-k) + S(j,r-k-1).$$

Now, by (3.6) and the usual recurrence for binomial coefficients, we have

$$S(i,j;n) = \sum_{r=0}^{i+j} \binom{n+1}{r+1} \mathcal{S}_r(i,j) = \sum_{r=0}^{i+j} \left\{ \binom{n}{r} + \binom{n}{r+1} \right\} \mathcal{S}_r(i,j)$$

$$= \sum_{r=0}^{i+j} \binom{n}{r} \mathcal{S}_r(i,j) + \sum_{r=1}^{i+j+1} \binom{n}{r} \mathcal{S}_{r-1}(i,j)$$

$$= \sum_{r=0}^{i+j+1} \binom{n}{r} \left\{ \mathcal{S}_r(i,j) + \mathcal{S}_{r-1}(i,j) \right\}.$$

However, $\mathcal{S}_r(i,j) + \mathcal{S}_{r-1}(i,j)$

$$= \sum_{k=0}^{r} k!\,(r-k)!\,S(i,k)S(j,r-k) + \sum_{k=0}^{r-1} k!\,(r-1-k)!\,S(i,k)S(j,r-1-k)$$

$$= \sum_{k=0}^{r} k!\,(r-k-1)!\,S(i,k)(r-k)S(j,r-k) + \sum_{k=0}^{r-1} k!\,(r-1-k)!\,S(i,k)S(j,r-1-k)$$

$$= \sum_{k=0}^{r} k!\,(r-1-k)!\left\{S(i,k)S(j+1,r-k) - S(i,k)S(j,r-k-1)\right\}, \quad \text{by (7.2)}$$
$$+ \sum_{k=0}^{r-1} k!\,(r-1-k)!\,S(i,k)S(j,r-1-k)$$

$$= \sum_{k=0}^{r-1} k!\,(r-k-1)!\,S(i,k)S(j+1,r-k) + r!\,S(i,r)S(j,0).$$

The extra term here may be dropped when we consider $j \geq 1$. Therefore, we have the new result that

$$S(i,j;n) = \sum_{r=0}^{i+j} \binom{n}{r+1} \sum_{k=0}^{r} k!\,(r-k)!\,S(i,k)S(j+1,r+1-k), \quad j \geq 1,\ i \geq 0. \tag{7.3}$$

Examples: Let $j = 1$ again. We find

$$S(0,1;n) = \binom{n}{1} + \binom{n}{2},$$

$$S(1,1;n) = \binom{n}{2} + \binom{n}{3},$$

$$S(2,1;n) = \binom{n}{2} + 3\binom{n}{3} + 2\binom{n}{4},$$

For $j = 1$, the general pattern of these coefficients begins as follows:

	0	1	2	3	4	5	6	7	$\cdots$	r
0	1	1								
1		1	1							
2		1	3	2						
3		1	7	12	6					
4		1	15	50	60	24				
5		1	31	180	390	360	120			
6		1	63	602	2100	3360	3520	720		
$\vdots$										
i										

It is interesting to note that these coefficients appear in another old formula:

$$S(i,0;n) = \sum_{k=0}^{i} (-1)^k \binom{n}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^i, \tag{7.4}$$

valid for $i \geq 1$, $n \geq 1$.

Examples:

$$S(1,0;n) = \binom{n}{1} + \binom{n}{2},$$

$$S(2,0;n) = \binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3},$$

$$S(3,0;n) = \binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4},$$

and so forth.

There is yet another old formula involving Stirling numbers of the second kind which we should mention. It is

$$S(i,0;n) = \sum_{r=0}^{i} (-1)^{i-r} \binom{n+r}{r+1} r! S(i,r), \quad n \geq 0, \ i \geq 1. \tag{7.5}$$

This occurs, for example, as the solution to a problem [13] in the *American Mathematical Monthly*.

Examples:

$$S(1,0;n) = \binom{n+1}{2},$$

$$S(2,0;n) = -\binom{n+1}{2} + 2\binom{n+2}{3},$$

$$S(3,0;n) = \binom{n+1}{2} - 6\binom{n+2}{3} + 6\binom{n+3}{4},$$

$$S(4,0;n) = -\binom{n+1}{2} + 14\binom{n+2}{3} - 36\binom{n+3}{4} + 24\binom{n+4}{5},$$

and so forth.

8. FINAL REMARKS

It is interesting to note that the original sum (1.1) is a type of convolution. So also formulas (3.6), (5.3), and (7.3) involve convolutions of the Stirling and Eulerian numbers. The formula found in [9] is not of this type. This is so because of the way in which the binomial theorem was first used. It would evidently be possible to obtain convolutions of the Bernoulli numbers. To get such a formula using Bernoulli polynomials is easy. Let us recall that

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x), \quad |t| < 2\pi, \tag{8.1}$$

defines the Bernoulli polynomial $B_n(x)$. Then $B_n(0) = B_n$ are the Bernoulli numbers. It is also a well-known old formula that then for all real x,

$$x^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k(x), \quad n \geq 0. \tag{8.2}$$

Form the product $k^i(n - k)^j$ by using this formula to expand k^i and $(n - k)^j$. Sum both sides and we get

$$(8.3)\qquad S(i,j;n) = \frac{1}{i+1}\sum_{r=0}^{i}\binom{i+1}{r}\frac{1}{j+1}\sum_{s=0}^{j}\binom{j+1}{s}\sum_{k=0}^{n}B_r(k)B_s(n-k),$$

which brings in a convolution of Bernoulli polynomials. Since the Bernoulli polynomials may be expressed in terms of Bernoulli numbers by the further formula

$$(8.4)\qquad B_n(x) = \sum_{m=0}^{n}\binom{n}{m}x^{n-m}B_m,$$

it would be possible to secure a convolution of the Bernoulli numbers. However, the author has not reduced this to any interesting or useful formula that appears to offer any advantages over those we have derived here or those in [9]. We leave this as a project for the reader.

It is also possible to obtain a mixed formula by proceeding first as in [9] to get

$$S(i,j;n) = \sum_{r=0}^{j}(-1)^r\binom{j}{r}n^{j-r}\sum_{k=0}^{n}k^{i+r},$$

apply one of our Stirling number expansions to the inner sum and get, e.g.,

$$(8.5)\qquad S(i,j;n) = \sum_{r=0}^{j}(-1)^r\binom{j}{r}n^{j-r}\sum_{k=0}^{i+r}\binom{n+1}{k+1}k!S(i+r,k),$$

but the writer sees no remarkable advantages to be gained.

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b-ADIC NUMBERS IN PASCAL'S TRIANGLE MODULO b

HEIKO HARBORTH
Technische Universität Braunschweig, West Germany

For the binomial coefficients in Pascal's triangle we write their smallest nonnegative residues modulo a base b. Then blocks of consecutive integers within the rows may be interpreted as b-adic numbers. What b-adic numbers can occur in the Pascal triangle modulo b? In this article we will give the density of such numbers and determine the smallest positive integer $h(b)$, such that its b-adic representation does not occur (see [3] for $b = 2$).

We use the notation

$$t = \sum_{i=0}^{m}a_ib^i = (a_ma_{m-1}\,\ldots\,a_1a_0)_b,\ 0 \le a_i \le b-1,\ a_m \ne 0,$$

for positive integers t. First we will prove the existence of b-adic numbers which do not occur.