Evidently, if $p \nmid h$, then

(4.1) $$(p^2;h) \equiv 0 \pmod{p^2};$$

otherwise,

(4.2) $$(p^2;h)/p \equiv (-1)^{h-1}p/h \pmod{p}.$$

We have, of course,

(4.3) $$(p^2;0) = 1 = (p^2;p^2).$$

As an application of the lemma, we have, for example:

(i) when $1 \leq m \leq 4$,

(4.4) $$S(p^2,m) \equiv \sum_{j \geq 0} (p^2;m + 5j) \pmod{p^2}.$$

On the right of the sigma in (4.4), we need consider only those nonnegative values of j for which

$$m + 5j \leq p^2 \text{ and } m + 5j \equiv 0 \pmod{p};$$

(ii) when $m = 0$, we have,

(4.5) $$S(p^2,0) - 1 \equiv \sum_{j \geq 1} (p^2;5j) \pmod{p^2},$$

so that

(4.6) $$\frac{S(p^2,0) - 1}{p} \equiv \sum_{j} (-1)^{j-1}/5j \pmod{p},$$

where $1 \leq j < p/5$. Thus

$$\frac{F_{121} - 1}{11} \equiv \frac{1}{5} - \frac{1}{10} + \frac{1}{4} - \frac{1}{9} \equiv 9 - 10 + 3 - 5 \equiv 8 \pmod{11}.$$

Therefore,

$$F_{121} \equiv 89 \pmod{121}.$$

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OPERATIONAL FORMULAS FOR UNUSUAL FIBONACCI SERIES

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Operational formulas can play a fascinating role in finding transformations and sums of series. For instance, by using the differential operator D $(=d/dx)$ we can transform

(1) $$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}, \quad |x| < 1,$$

into

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1 - x)^2}, \quad |x| < 1.$$

The operator $\theta = xD$ is even more interesting. It has the basic property that $\theta^p x^k = k^p x^k$, so that (1) can be transformed into

$$\sum_{k=0}^{\infty} k^p x^k = \theta^p \left\{\frac{1}{1-x}\right\}, \tag{2}$$

and since it can also be shown (and is well known) that

$$\theta^p f(x) = \sum_{k=0}^{p} S(p,\ k) x^k D^k f(x), \tag{3}$$

where $S(p,\ k)$ are Stirling numbers of the second kind, explicitly

$$k!S(p,\ k) = \Delta^k 0^p = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^p, \tag{4}$$

then series (2) can be found in closed form for it is trivial to find the higher derivatives needed in (3). The result is a very old and well-known formula. In [7] is given an extension of (3) applied to generalized Hermite polynomials. There are numerous similar generalized expansions involving the D operator. Here we propose to examine some rather unusual variations that are not too well known, and which have applications to Fibonacci numbers among other things.

We shall need several other well-known operational formulas whose proofs involve some calculus and/or mathematical induction, and we tabulate these below:

$$\theta = D_z, \text{ where } x = e^z, \tag{5}$$

$$\theta^n = D_z^n, \tag{6}$$

$$x^n D_x^n = n!\binom{D_z}{n}, \tag{7}$$

where the binomial coefficient is defined as usual by $\binom{x}{n} = x(x-1)\ \ldots\ (x-n+1)/n!$, with $\binom{x}{0} = 1$.

$$e^D = 1 + \Delta = E, \tag{8}$$

where

$$\Delta f(x) = f(x+1) - f(x) \quad \text{and} \quad Ef(x) = f(x+1).$$

More generally

$$e^{tD_x} = f(x+t) = \underset{x,h}{E} f(x). \tag{9}$$

The q-operator

$$f(qx) = Qf(x), \text{ where } Q = q^{\theta}. \tag{10}$$

This was used, e.g., in [10], and is very convenient when working with basic hypergeometric series.

In the references at the end are several papers, viz. [1], [2], [4], [5], from the older literature where properties of a great number of familiar and unfamiliar operators were developed. The master calculator was almost certainly George Boole. The English literature for the period from about 1830 to 1890 is especially rich in papers on unusual operators.

In [1], Boole gave the pair of very remarkable operational expansions

$$f(x + \phi'(D))u(x) = e^{\phi(D)} f(x) e^{-\phi(D)} u(x), \tag{11}$$

and

$$f(D + \phi'(x))u(x) = e^{-\phi(x)} f(D) e^{\phi(x)} u(x), \tag{12}$$

which hold for arbitrary functions f, $\emptyset$, and u. The formulas are certainly true for polynomials, and in order to avoid matters of convergence of any series we shall explain that we interpret these as statements about formal power series. In that context there is no difficulty and we use formal power series definitions of all operators. Thus, if L is a linear operator, we should like to define e^L by

$$e^L = \sum_{k=0}^{\infty} \frac{1}{k!} L^k. \tag{13}$$

Boole's formulas (11)-(12) have a bearing on expansions in [7]. They are representative of some of the most unusual operational formulas.

But stranger still, we shall consider the operator L^L, which we define as follows:

$$\begin{aligned} L^L f(x) &= \left\{(L - 1) + 1\right\}^L f(x) = \sum_{n=0}^{\infty} \binom{L}{n} (L - 1)^n f(x) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} C_j^n L^j \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} L^k f(x) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} C_j^n L^{j+k} f(x), \end{aligned} \tag{14}$$

where C_j^n are Stirling numbers of the first kind, i.e., coefficients in the expansion of a binomial coefficient:

$$\binom{x}{n} = \sum_{j=0}^{n} C_j^n x^j. \tag{15}$$

In the familiar notation of Riordan, $n! C_j^n = s(n, j)$.

For a particular choice of L we may be able to give a more compact definition. Thus, with $f = f(x)$,

$$\begin{aligned} D^D f &= \left\{(D - 1) + 1\right\}^D f = \sum_{n=0}^{\infty} \binom{D}{n} (D - 1)^n f \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} D_z^n (D_x - 1)^n f, \text{ by (7), with } z = e^x, \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} D_z^n D_x^k f(x). \end{aligned} \tag{16}$$

For an example of this expansion, let $f(x) = e^{ax}$. Then

$$D_x^k e^{ax} = a^k e^{ax} = a^k z^a,$$

whence

$$\begin{aligned} D^D e^{ax} &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} D_z^n (a^k z^a) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^k \binom{a}{n} n! z^{a-n} \\ &= z^a \sum_{n=0}^{\infty} \binom{a}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^k \end{aligned}$$

$$= z^a \sum_{n=0}^{\infty} \binom{a}{n} (a-1)^n = z^a a^a = a^a e^{ax},$$

so that we have the attractive formula

$$D^D e^{ax} = a^a e^{ax}. \tag{17}$$

It is instructive to compare this with $\theta^p x^k = k^p x^k$, and to recall a little terminology from vector analysis. A characteristic vector for a linear transformation L is a non-zero vector f such that $Lf = cf$ for some scalar c. With each operator we like to find a natural function or characteristic function. For the operator θ it is x^k, for D^D it is e^{ax}, etc.

Formula (17) allows us to write down symbolic sums for various peculiar series. Thus

$$\sum_{k=0}^{n-1} k^k e^{kx} t^k = \sum_{k=0}^{n-1} t^k D^D e^{kx} = D^D \sum_{k=0}^{n-1} (te^x)^k = D^D \left\{ \frac{t^n e^{nx} - 1}{te^x - 1} \right\}. \tag{18}$$

In particular,

$$\sum_{k=0}^{\infty} k^k t^k = D^D \left\{ \frac{1}{1 - te^x} \right\} \Bigg|_{x=0}. \tag{19}$$

All that would be necessary to sum (19) would be to find a different method of attaching a meaning to the right-hand member.

For a Fibonacci-Lucas application, recall the general Lucas function

$$L_n = L_n(a, b) = a^n + b^n.$$

Then

$$\sum_{k=0}^{n-1} k^k e^{kx} L_{pk} = D^D \left\{ \frac{a^{pn} e^{nx} - 1}{a^p e^x - 1} + \frac{b^{pn} e^{nx} - 1}{b^p e^x - 1} \right\}, \tag{20}$$

and for the generalized Fibonacci function

$$F_n = F_n(a, b) = (a^n - b^n)/(a - b),$$

then

$$(a - b) \sum_{k=0}^{n-1} k^k e^{kx} F_{pk} = D^D \left\{ \frac{a^{pn} e^{nx} - 1}{a^p e^x - 1} - \frac{b^{pn} e^{nx} - 1}{b^p e^x - 1} \right\} \tag{21}$$

Following the methods outlined in [3], [6], [8], [9], or [11], we could set down complicated symbolic formulas for the general series

$$\sum_{k=0}^{n-1} k^k e^{kx} u^k F_{pk}^r L_{qk}^s, \tag{22}$$

but we shall not take the space to exhibit the result.

For another application, let us rewrite (17) as $a^a = e^{-x} D^D e^{ax}$, so that we have an obvious application in the two forms

$$L_n^{L_n} = e^{-x} D^D e^{L_n x}, \quad F_n^{F_n} = e^{-x} D^D e^{F_n x}, \tag{23}$$

which allow us to introduce Fibonacci powers of Fibonacci (and Lucas powers of Lucas) numbers into known series. In particular,

$$\sum_{n=0}^{\infty} t^n L_n^{L_n} = e^{-x} D^D \sum_{n=0}^{\infty} t^n e^{L_n x}, \tag{24}$$

and a similar formula with F in place of L.

In principle then we could sum such series if we could sum the series

$$S(t, u) = \sum_{n=0}^{\infty} t^n u^{L_n}, \tag{25}$$

and

$$T(t,\ u) = \sum_{n=0}^{\infty} t^n u^{F_n}. \tag{26}$$

These are offered as research projects; the author would be interested in hearing of any success by others. $D_u S|_{u=1}$ and $D_u T|_{u=1}$ are known.

The operator θ^θ may be considered finally. We find

$$\theta^\theta f = \{(\theta - 1) + 1\}^\theta f = \sum_{n=0}^{\infty} \binom{\theta}{n} (\theta - 1)^n f$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} D_x^n (\theta - 1)^n f, \text{ by (7)},$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} D_x^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \theta^k f,$$

so we have

$$\theta^\theta f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} D_x^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \theta^k f(x). \tag{27}$$

Let $f(x) = x^p$, then

$$\theta^\theta x^p = \sum_{n=0}^{\infty} \frac{x^n}{n!} D^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} p^k x^p = \sum_{n=0}^{\infty} \frac{x^n}{n!} (p - 1)^n \binom{p}{n} n! x^{p-n}$$

$$= x^p \sum_{n=0}^{\infty} \binom{p}{n} (p - 1)^n = p^p x^p,$$

or therefore, another formula analogous to (17),

$$\theta^\theta x^p = p^p x^p. \tag{28}$$

As an application we can get a different version of formula (19) as

$$\sum_{k=0}^{\infty} k^k x^k = \theta^\theta \sum_{k=0}^{\infty} x^k = \theta^\theta \left\{ \frac{1}{1 - x} \right\}. \tag{29}$$

We wish to remark that even stranger formulas have been published. Cayley [4], [5] expressed the Lagrange series inversion formula in the most curious operational form

$$F(x) = (D_u)^{hD_h - 1} \{F'(u) e^{hf(u)}\}, \tag{30}$$

where $x = u + hf(x)$ and $F(x)$ is an arbitrary function. By differentiation, he expressed the second form of this expansion as

$$\frac{F(x)}{1 - hf'(x)} = (D_u)^{hD_h} \{F(u) e^{hf(u)}\}. \tag{31}$$

Cayley says these are well known, and goes on to write similar formulas for functions of several variables.

Bronwin [2] writes

$$f(a + x) = D_a^\theta \{f(a) e^x\} \tag{32}$$

as a symbolic form of Taylor's expansion. This is, of course, a special case of the Lagrange expansion.

In conclusion, we wish to emphasize that the formulas presented here are offered more for further research than as final answers to any of the questions raised. It certainly is possible to introduce unusual terms into generating functions by the use of unusual operators.

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