Form the product $k^i(n - k)^j$ by using this formula to expand k^i and $(n - k)^j$. Sum both sides and we get

$$(8.3) \qquad S(i,j;n) = \frac{1}{i+1}\sum_{r=0}^{i}\binom{i+1}{r}\frac{1}{j+1}\sum_{s=0}^{j}\binom{j+1}{s}\sum_{k=0}^{n} B_r(k)B_s(n-k),$$

which brings in a convolution of Bernoulli polynomials. Since the Bernoulli polynomials may be expressed in terms of Bernoulli numbers by the further formula

$$(8.4) \qquad B_n(x) = \sum_{m=0}^{n}\binom{n}{m}x^{n-m}B_m,$$

it would be possible to secure a convolution of the Bernoulli numbers. However, the author has not reduced this to any interesting or useful formula that appears to offer any advantages over those we have derived here or those in [9]. We leave this as a project for the reader.

It is also possible to obtain a mixed formula by proceeding first as in [9] to get

$$S(i,j;n) = \sum_{r=0}^{j}(-1)^r\binom{j}{r}n^{j-r}\sum_{k=0}^{n}k^{i+r},$$

apply one of our Stirling number expansions to the inner sum and get, e.g.,

$$(8.5) \qquad S(i,j;n) = \sum_{r=0}^{j}(-1)^r\binom{j}{r}n^{j-r}\sum_{k=0}^{i+r}\binom{n+1}{k+1}k!S(i+r,k),$$

but the writer sees no remarkable advantages to be gained.

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b-ADIC NUMBERS IN PASCAL'S TRIANGLE MODULO b

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For the binomial coefficients in Pascal's triangle we write their smallest nonnegative residues modulo a base b. Then blocks of consecutive integers within the rows may be interpreted as b-adic numbers. What b-adic numbers can occur in the Pascal triangle modulo b? In this article we will give the density of such numbers and determine the smallest positive integer $h(b)$, such that its b-adic representation does not occur (see [3] for $b = 2$).

We use the notation

$$t = \sum_{i=0}^{m} a_i b^i = (a_m a_{m-1} \ldots a_1 a_0)_b, \; 0 \le a_i \le b - 1, \; a_m \ne 0,$$

for positive integers t. First we will prove the existence of b-adic numbers which do not occur.

Lemma 1: $(1011)_2$ is not to be found within any row of the Pascal triangle modulo 2.

Proof: We assume that there are integers n and k with

$$\binom{n}{k} \equiv \binom{n}{k+2} \equiv \binom{n}{k+3} \equiv 1 \quad \text{and} \quad \binom{n}{k+1} \equiv 0 \pmod{2}.$$

These congruences substituted in

$$(1) \qquad (k+1+i)\binom{n}{k+1+i} = (n-k-i)\binom{n}{k+i}$$

for $i = 0, 1, 2$, gives $n \equiv k \pmod{2}$, $k \equiv 0 \pmod{2}$, and $n \equiv 1 \pmod{2}$, respectively, which is a contradiction.

Lemma 2: $(111)_b$ is not to be found within any row of Pascal's triangle modulo b with $b > 2$.

Proof: We assume that

$$\binom{n}{k} \equiv \binom{n}{k+1} \equiv \binom{n}{k+2} \equiv 1 \pmod{b}.$$

Together with (1), for $i = 0$ and $i = 1$, we conclude that $n \equiv 2k + 1 \pmod{b}$, and $n \equiv 2k + 3 \pmod{b}$, respectively. However, both congruences are possible only if $b = 2$.

We are now able to determine the density.

Theorem 1: Almost all b-adic numbers cannot occur within the rows of Pascal's triangle modulo b.

Proof: As noted in [4], it is well known that the density of those b-adic integers not containing a given sequence of digits is 0 (see [2], p. 120). Thus, the proof is given by Lemmas 1 and 2.

Theorem 2: Let $h(b)$ be the smallest b-adic number not being found within any row of Pascal's triangle modulo b. Then, $h(b) = b^2 + b + 1 = (111)_b$ for $b > 2$, and $h(2) = 11 = (1011)_2$.

We first prove two lemmas.

Lemma 3: Let $b = b_1b_2$ with $(b_1, b_2) = 1$. Then $(a_m \ldots a_0)_b$ occurs in the Pascal triangle modulo b if and only if $(a_{im} \ldots a_{i0})_{b_i}$ for $i = 1, 2$ occur in the triangles modulo b_i with $a_{ij} \equiv a_j \pmod{b_i}$, $j = 0, 1, \ldots, m$.

Proof: One direction of the proof is trivial.

In the following, we use the result of [1] and [6], that $\binom{n}{k} \pmod{b}$ is periodic for fixed k with the minimal period N being the product of all prime powers $p^{\alpha+\beta}$ with p^{α} from the canonical factorization of b and β from $p^{\beta} \le k < p^{\beta+1}$. Thus, N depends only on the prime factors of b and on k (see [5] for further references). By reasons of symmetry, a corresponding periodicity of length L holds for $\binom{n+\ell}{k+\ell}$ with fixed n and k.

From this and by the assumption, we are able to find n_i and k_i such that for $i = 1, 2$,

$$\binom{n_i}{k_i+j} \equiv \binom{n_i + x_iL_i}{k_i + x_iL_i + j} \equiv a_{i(m-j)} \pmod{b_i}, \; j = 0, 1, \ldots, m,$$

with minimal periods L_i each being the lowest common multiple of $m + 1$ minimal periods. From $(b_1, b_2) = 1$ we have $(L_1, L_2) = 1$. Thus, the diophantine equation,

$$k_1 + x_1L_1 = k_2 + x_2L_2,$$

has solutions x_1, x_2. For fixed values x_1, x_2, we then have minimal periods N with

$$\binom{n_i + x_iL_i + y_iN_i}{k_i + x_iL_i + j} \equiv a_{i(m-j)} \pmod{b_i},\ j = 0, 1, \ldots, m.$$

Finally, $(N_1, N_2) = 1$ guarantees solutions y_1, y_2 of

$$n_1 + x_1L_1 + y_1N_1 = n_2 + x_2L_2 + y_2N_2,$$

which completes the proof.

Lemma 4: In Pascal's triangle modulo p^α, p being a prime, there are arbitrarily large partial triangles with

$$\binom{n + n_r}{k + k_r} \equiv r\binom{n}{k} \pmod{p^\alpha},\ n \geq 0,\ k \geq 0,$$

for every r from 1 to p^α.

Proof: We first show

$$\binom{rp^{\alpha\beta}}{k} \equiv 0 \pmod{p^\alpha} \text{ for } p^{\alpha\beta} - p^{\alpha\beta - \alpha + 1} < k < p^{\alpha\beta} + p^{\alpha\beta - \alpha + 1},\quad k \neq p^{\alpha\beta}. \tag{2}$$

Let γ be the exponent of p in the canonical factorization of the binomial coefficient in (2). Then, by a theorem of Legendre ([7], p. 13), we have,

$$\begin{aligned}\gamma &= \sum_{i \geq 1}\left\{\left[\frac{rp^{\alpha\beta}}{p^i}\right] - \left[\frac{k}{p^i}\right] - \left[\frac{rp^{\alpha\beta} - k}{p^i}\right]\right\}\\ &\geq \sum_{i=1}^{\alpha\beta}\left\{-\left[\frac{k}{p^i}\right] - \left[\frac{-k}{p^i}\right]\right\} \geq \sum_{i=\alpha\beta-\alpha+1}^{\alpha\beta} 1 = \alpha,\end{aligned}$$

where $[x]$ means the greatest integer not exceeding x.

We further show by induction on α that $\binom{rp^{\alpha\beta}}{p^{\alpha\beta}}$, for $r = 1, 2, \ldots, p^\alpha$, is a complete system of residues modulo p^α. Let

$$P_j(r) = \prod_{\substack{i=1\\(i,p)=1}}^{p^j - 1} \frac{(r-1)p^j + i}{i}. \tag{3}$$

Then for $\alpha = 1$ we can write

$$\binom{rp^\beta}{p^\beta} = r\prod_{j=1}^{\beta} P_j(r) \equiv r \pmod{p}.$$

In general, with $r = vp^{\alpha-1} + \rho$, $1 \leq \rho \leq p^{\alpha-1}$, $0 \leq v \leq p - 1$, we get

$$\binom{rp^{\alpha\beta}}{p^{\alpha\beta}} = r\prod_{j=1}^{\alpha\beta} P_j(r) \equiv r\prod_{j=1}^{\alpha-1} P_j(r) \equiv r\prod_{j=1}^{\alpha-1} P_j(\rho) \equiv vp^{\alpha-1} + \rho\prod_{j=1}^{\alpha-1} P_j(\rho) \pmod{p^\alpha}.$$

If we assume $\rho\pi P_j(\rho)$ to take all residues modulo $p^{\alpha-1}$, then the induction is complete.

As β may be chosen arbitrarily large, Lemma 4 follows with $n_r = rp^{\alpha\beta}$ and $k_r = p^{\alpha\beta}$.

Proof of Theorem 2: Lemmas 1 and 2 yield $h(b) \leq \ldots$. Because of Lemma 3, we need to consider only prime powers as moduli. Trivially,

$$(a_0)_{p^\alpha}, \; 1 \leq a_0 < p^\alpha,$$

occur as $\binom{n}{k}$ in the Pascal triangle modulo p^α (let $n = a_0$ and $k = 1$), and so do $(1a_o)_{p^\alpha}$ (let $n = a_o$ and $k = 0, 1$), with $1 \leq a_0 \leq p^\alpha$. We then multiply the digits of $(1a_0)_{p^\alpha}$ by r, $1 \leq r < p^\alpha$, and obtain all numbers $(a_1a_0)_{p^\alpha}$, including those with $(a_1, p^\alpha) > (a_0, p^\alpha)$. This is because of Lemma 4 and the symmetry of binomial coefficients. Further, $(100)_{p^\alpha}$ occurs if $n = 2p^\alpha$, $k = 0$, 1, 2, and $(110)_{p^\alpha}$ if $n = 2p^\alpha + 1$, $k = 0$, 1, 2.

Now

$$\sum_{i \geq 1}\left\{\left[\frac{rp^\alpha - 2}{p^i}\right] - \left[\frac{p^\alpha - 1}{p^i}\right] - \left[\frac{(r-1)p^\alpha - 1}{p^i}\right]\right\} \geq \sum_{i=1}^{\alpha}\{\cdots\} = \alpha,$$

so that $\binom{n}{k} \equiv 0 \pmod{p^\alpha}$, if $n = rp^\alpha - 2$ and $k = p^\alpha - 1$. Using (3), and with v being an integer, we have

$$\binom{rp^\alpha - 2}{p^\alpha - 2} = \frac{p^\alpha - 1}{rp^\alpha - 1}\prod_{j=1}^{\alpha} P_j(r) \equiv 1 + vp \pmod{p^\alpha}, \tag{4}$$

$$\binom{rp^\alpha - 2}{p^\alpha} = (r-1)\frac{(r-1)p^\alpha - 1}{p^\alpha - 1}\binom{rp^\alpha - 2}{p^\alpha - 2} \equiv (r-1)\binom{rp^\alpha - 2}{p^\alpha - 2} \pmod{p^\alpha}. \tag{5}$$

As $(1 + vp, p^\alpha) = 1$, we can find an integer x such that multiplying (4) and (5) by x yields the residues 1 and $r - 1$. Because of Lemma 4, corresponding binomial coefficients occur in the Pascal triangle, so that the existence of all numbers $(10(r-1))_{p^\alpha}$, $2 \leq r \leq p^\alpha$, is proved.

Thus, we have shown $h(b) \geq (111)_b$ for $b \geq 2$. The remaining binary numbers $(111)_2$, $(1000)_2$, $(1001)_2$, and $(1010)_2$ are to be found within the rows 3, 4, 5, and 6, respectively.

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