ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS *to* Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. *This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.*

H-295 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA.*

Establish the identities

(a) $F_k F_{k+6r+3}^2 - F_{k+8r+4}^2 F_{k+2r+1} = (-1)^{k+1} F_{2r+1}^3 L_{2r+1} L_{k+4r+2}$

and

(b) $F_k F_{k+6r}^2 - F_{k+8r} F_{k+2r}^2 = (-1)^{k+1} F_{2r}^3 L_{2r} L_{k+4r}$.

H-296 *Proposed by C. Kimberling, University of Evansville, Evansville, IN.*

Suppose x and y are positive real numbers. Find the least positive integer n for which

$$\left[\frac{x}{n+y}\right] = \left[\frac{x}{n}\right]$$

where $[z]$ denotes the greatest integer less than or equal to z.

H-297 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.*

Let $P_0 = P_1 = 1$, $P_n(\lambda) = P_{n-1}(\lambda) - \lambda P_{n-2}(\lambda)$. Show

$$\lim_{n\to\infty} P_{n-1}(\lambda)/P_n(\lambda) = (1 - \sqrt{1-4\lambda})/2\lambda = \sum_{n=0}^{\infty} C_{n+1}x^n,$$

where C_n is the nth Catalan number. Note that the coefficients of $P_n(\lambda)$ lie along the rising diagonals of Pascal's triangle with alternating signs.

H-298 *Proposed by L. Kuipers, Mollens, Valais, Switzerland.*

Prove:

(i) $F_{n+1}^6 - 3F_{n+1}^5 F + 5F_{n+1}^3 F_n^3 - 3F_{n+1}F_n^5 - F_n^6 = (-1)^n$, $n = 0, 1, \ldots;$

(ii) $F_{n+6}^6 - 14F_{n+5}^6 - 90F_{n+4}^6 + 350F_{n+3}^6 - 90F_{n+2}^6 - 14F_{n+1}^6 + F_n^6$

$= (-1)^n 80$, $n = 0, 1, \ldots;$

(iii) $F_{n+6}^6 - 13F_{n+5}^6 + 41F_{n+4}^6 - 41F_{n+3}^6 + 13F_{n+2}^6 - F_{n+1}^6$

$\equiv -40 + \frac{1}{2}(1 + (-1)^n)80 \pmod{144}$.

SOLUTIONS

A Soft Matrix

H-274 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX.*

It has been shown [*The Fibonacci Quarterly* 2, No. 3 (1964):261-266] that

$$\text{if } Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \text{ then } Q^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1} - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix}.$$

Generalize the matrix Q to solutions of the difference equation

$$U_n = rU_{n-1} + sU_{n-2},$$

where r and s are arbitrary real numbers, $U_0 = 0$ and $U_1 = 1$.

Solved by the proposer.

The key to the extension is the identity

$$F_{n+1}^2 - F \quad F \quad = F^2 + F \quad F_{n+1},$$

which allows one to generalize the central entry of Q . It is easily established then by mathematical induction that

$$\text{if } R = \begin{pmatrix} 0 & 0 & s^2 \\ 0 & s & 2rs \\ 1 & r & r^2 \end{pmatrix}, \text{ then } R^n = \begin{pmatrix} s^2U_{n-1}^2 & s^2U_{n-1}U_n & s^2U_n^2 \\ 2sU_{n-1}U & s(U_n^2 + U_{n-1}U_{n+1}) & 2sU_nU_{n+1} \\ U^2 & U_nU_{n+1} & U_{n+1}^2 \end{pmatrix}.$$

A Corrected Oldie

H-225 *Proposed by G. A. R. Guillotte, Quebec, Canada.*

Let p denote an odd prime and $x^p + y^p = z^p$ for positive integers x, y, and z. Show that

A) $p < x/(z - x) + y/(z - y)$

and

B) $z/2(z - x) < p < y/(z - y)$.

Solved by the proposer.

Consider $(x/z)^i + (y/z)^i = 1 + \varepsilon_i$ for $\varepsilon_0 = 1$, $\varepsilon_p = 0$, and $\varepsilon_i \in (0, 1)$, for $1 \le i \le p - 1$. Then

$$\sum_{i=0}^{p} (x/z)^i + \sum_{i=0}^{p} (y/z)^i = p + 1 + \sum_{i=0}^{p} \varepsilon_i$$

becomes

$$(1 - (x/z)^{p+1})/(1 - x/z) + (1 = (y/z)^{p+1})/(1 - y/z) = p + 1 + \sum_{i=0}^{p} \varepsilon_i .$$

Now

$$1/(1 - x/z) + 1/(1 - y/z) > p + 1 + \sum_{i=0}^{p} \varepsilon_i .$$

Hence

$$z/(z - x) + z/(z - y) > p + 1 + 1 + \sum_{i=1}^{p-1} \varepsilon_i ,$$

since $\varepsilon_0 = 1$ and $\varepsilon_p = 0$. But

$$z/(z - x) - 1 = x/(z - x) \text{ and } z/(z - y) - 1 = y/(z - y).$$

Therefore

$$x/(z - x) + y/(z - y) > p + \sum_{i=1}^{p-1} \varepsilon_i > p.$$

Similar reasoning leads to part B).

Editorial Note: Please keep working on those oldies!

Special Note: It has long been known that any solution for the basic pair of equations for 103 as a *congruent number* would entail enormous numbers. For that reason, 103 had not been proved congruent: on the other hand, it had not been proved noncongruent.

Then, in 1975, two brilliant computer experts—Dr. Katelin Gallyas and Mr. Michael Buckley—finally proved 103 to be congruent, working along lines suggested by J. A. H. Hunter. The big IBM 370 computer of the University of Toronto was used for this achievement.

For the system

$$X^2 - 103Y^2 = Z^2, \; X^2 + 103Y^2 = W^2,$$

the minimal solution was found to be:

X = 134 13066 49380 47228 37470 20010 79697
Y = 7 18866 17683 65914 78844 74171 61240
Z = 112 55362 67770 44455 63954 40707 12753
W = 152 68841 36166 82668 99188 22379 29103

REFERENCE

"Fibonacci Newsletter," September 1975.

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