# MATRIX GENERATORS OF PELL SEQUENCES

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## SECTION 1

The Pell sequence $\{P_n\}$ is defined recursively by the equation

(1) $$P_{n+1} = 2P_n + P_{n-1},$$

$n = 2, 3, \ldots$, where $P_1 = 1$, $P_2 = 2$. As is well known (see, e.g., [1]), the members of this sequence are also generated by the matrix

$$M = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix},$$

since by taking successive positive powers of $M$ one can easily establish that

$$M^n = \begin{vmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{vmatrix}.$$

Related to the sequence $\{P_n\}$ is the sequence $\{R_n\}$, which is defined recursively [1] by

$$R_{n+1} = 2R_n + R_{n-1},$$

$n = 2, 3, \ldots$, $R_1 = 2$, $R_2 = 6$. In what follows, we will require two other Pell sequences; they are best motivated by considering the following problem (cp. [2]): do there exist sequences $\{p_n\}$, $p_1 = 1$, satisfying (1) which are also "geometric" (i.e., the ratio between terms is constant)? These two requirements are easily seen to be equivalent to $p_n$ satisfying the so-called "Pell equation" [1]:

(2) $$p^2 = 2p + 1.$$

The positive root of this equation is $\psi = \frac{1}{2}(2 + \sqrt{8})$, and one easily checks that the sequence $\{\psi^n\}$ is a "geometric" Pell sequence. In a similar manner, by considering the negative root in (2), $\psi' = \frac{1}{2}(2 - \sqrt{8})$, one obtains a second geometric Pell sequence $\{\psi'^n\}$. (Since $\psi' = \frac{-1}{\psi}$, these two sequences are by no means distinct. However, it will be convenient in what follows to consider them separately.) That these four sequences are related to each other is apparent from the following well-known Binet-type formulas, which are verified mathematically by induction [1]:

$$P_n = \frac{\psi^n - \psi'^n}{\psi - \psi'}, \quad R_n = \psi^n + \psi'^n, \quad \psi^n = \frac{1}{2}(R_n + P_n\sqrt{8}).$$

Our purpose in this paper is threefold: we will give a constructive method for finding all possible matrix generators of the above Pell sequences; we show that, in fact, all such matrices are naturally related to each other; and finally, by applying well-known results from matrix algebra, we establish the above Binet-type formulas and several other well-known Pell identities.

## SECTION 2

A direct calculation shows that the matrix

$$M = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$$

satisfies the Pell equation; i.e.,

$$M^2 = 2M + I,$$

where $I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. Let $A = \begin{vmatrix} x & y \\ u & v \end{vmatrix}$, where $x$, $y$, $u$, $v$ are to be determined subject only to the condition that $xv - yu \neq 0$. Substitution of $A$ into (2) results in the following system of scalar equations:

(3.1) $x^2 - 2x - 1 + yu = 0$

(3.2) $(x + v - 2)y = 0$

(3.3) $(x + v - 2)u = 0$

(3.4) $v^2 - 2v - 1 + yu = 0$

We now investigate possible solutions of these equations. Since the techniques are similar to those used in [3], we omit most of the details.

*Case 1: $y = 0$*

Equations (3.1), (3.4) reduce to the Pell equation, implying

$$x = \{\psi, \psi'\}, \quad v = \{\psi, \psi'\}.$$

(a) If $u = 0$, we obtain the following matrix generators:

$$\Psi_0 = \begin{vmatrix} \psi & 0 \\ 0 & \psi' \end{vmatrix}, \qquad \Psi_1 = \begin{vmatrix} \psi & 0 \\ 0 & \psi \end{vmatrix},$$

$$\Psi_2 = \begin{vmatrix} \psi' & 0 \\ 0 & \psi \end{vmatrix}, \qquad \Psi_3 = \begin{vmatrix} \psi' & 0 \\ 0 & \psi' \end{vmatrix}.$$

(b) If $u \neq 0$, (3.3) implies $x + v = 2$, and hence, that

$$\Psi_{0u} = \begin{vmatrix} \psi & 0 \\ u & \psi' \end{vmatrix}, \qquad \Psi_{2u} = \begin{vmatrix} \psi' & 0 \\ u & \psi \end{vmatrix}.$$

The $n$th power of the matrix $\Psi_{0u}$ is easily shown to be

$$\Psi_{0u}^n = \begin{vmatrix} \psi^n & 0 \\ P_n u & \psi'^n \end{vmatrix},$$

where $\{P_n\}$ is the sequence defined in (1).

*Case 2:* $y \neq 0$

(a) If $u = 0$, the situation is similar to that of Case 1(b), and we omit the details.

(b) Suppose $u \neq 0$. Equation (3.3) implies $x = 2 - v$—this is consistent with (3.2)—and substitution for $x$ in (3.1) gives, after collecting terms

$$v^2 - 2v - 1 + yu = 0,$$

which is consistent with (3.4). Thus, the assumptions $y \neq 0$, $u \neq 0$ result in the following reduced system of equations:

$$v = \frac{1}{2}(2 \pm \sqrt{8 - 4yu}) \tag{4.1}$$

$$x = 2 - v. \tag{4.2}$$

Before investigating some matrix generators corresponding to solutions of the equations (4.1), (4.2), we pause to summarize our results.

We have been tacitly assuming that for a matrix $A$ to be a generator of Pell sequences it must satisfy (2), the Pell equation. However, since our prototype generator is the matrix

$$M = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix},$$

whose characteristic equation is easily seen to be the Pell equation (2), and since this latter equation is also the minimal equation for $M$, we would like to restrict our matrices $A$ to those which also have the latter property. The initial assumption on $A$, $xv - yu \neq 0$, rules out, e.g., a matrix of the form

$$A = \begin{vmatrix} \psi & 0 \\ 0 & 0 \end{vmatrix},$$

which evidently satisfies (2). We would, however, also like to rule out matrices of the form $\Psi_1$ and $\Psi_3$ which satisfy (2) but do not have (2) as minimal equation. Thus, the following

*Definition:* A $2 \times 2$ matrix $A = \begin{vmatrix} x & y \\ u & v \end{vmatrix}$ is said to be a nontrivial generator of Pell sequences if $xv - yu \neq 0$, and its minimal equation is the Pell equation (2).

The above discussion then completely characterizes nontrivial generators of Pell sequences, which we summarize in the following:

*Theorem:* A $2 \times 2$ matrix $A$ is a nontrivial generator of Pell sequences if and only if it is similar to

$$\Psi_0 = \begin{vmatrix} \psi & 0 \\ 0 & \psi' \end{vmatrix}.$$

*Remark 1:* Evidently, $M$ is similar to $\Psi_0$. [We show below that $M$ is obtained as a nontrivial generator by an appropriate choice of solutions to the system (4.1), (4.2).] In light of this similarity an indirect way of obtaining nontrivial generators is to form the product $Q\Psi_0 Q^{-1}$, for any nonsingular matrix $Q$.

SECTION 3

*Example 1*: If we limit $y$, $u$ to be positive integer values in (4.1), then there is a unique pair which keeps the radicand positive: $y = u = 1$. This results in two sets of solutions:

$$y = 1, \quad u = 1, \quad v = 2, \quad x = 0$$

and

$$y = 1, \quad u = 1, \quad v = 0, \quad x = 2.$$

The latter set results in the "$M$-matrix"

$$M = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix},$$

where

$$M^n = \begin{vmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{vmatrix}.$$

(Cp. §1.) Since $M^n$ is similar to $\Psi_0^n$, we conclude that the traces and determinants of these two matrices are the same. Hence,

$$P_{n+1} + P_{n-1} = \psi^n + \psi'^n \tag{5}$$

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n, \tag{6}$$

two well-known Pell identities [1].

*Example 2*: In (4.1), take $y = 2$, $u = 1$. Then one obtains

$$N = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix},$$

and one easily checks that

$$N^n = \begin{vmatrix} \frac{1}{2}R_n & 2P_n \\ P_n & \frac{1}{2}R_n \end{vmatrix}.$$

Similarity of $N^n$ with $\Psi_0^n$ implies (trace invariance) that

$$R_n = \psi^n + \psi'^n \tag{7}$$

and that (determinant invariance)

$$R_n^2 - 8P_n^2 = 4(-1)^n. \tag{8}$$

Whereas, similarity of $N^n$ with $M^n$ implies, respectively (by trace and determinant invariance), that (cp. [1])

$$R_n = P_{n+1} + P_{n-1} \tag{9}$$

$$R_n^2 = 4(P_{n+1}P_{n-1} + P_n^2). \tag{10}$$

*Example 3:* In (4.1), take $y = 2$, $u = -1$; one possible set of solutions for $x$ and $v$ is, respectively, $x = 3$, $v = -1$, and we obtain

$$H = \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix},$$

$$H^n = \begin{vmatrix} -\frac{1}{2}R_{n-1} & 2P_n \\ -P_n & \frac{1}{2}R_{n+1} \end{vmatrix}.$$

Similarity of $H^n$ with $\Psi_0^n$ gives (cp. [1])

$$(11) \qquad R_{n+1} - R_{n-1} = 2(\psi^n + \psi'^n)$$

$$(12) \qquad 8P_n^2 - R_{n+1}R_{n-1} = 4(-1)^n.$$

*Note 1:* Lines (12) and (8) imply that

$$R_n^2 - R_{n+1}R_{n-1} = 8(-1)^n,$$

or

$$R_{n+1}R_{n-1} - R_n^2 = 8(-1)^{n+1}.$$

(Cp. [1].)

Similarity of $H^n$ with $M^n$ gives

$$(13) \qquad P_{n+1} + P_{n-1} = \frac{1}{2}(R_{n+1} - R_{n-1})$$

$$(14) \qquad R_{n+1}R_{n-1} = 4(3P_n^2 - P_{n+1}P_{n-1}).$$

Similarity of $H^n$ with $N^n$ gives (cp. [1])

$$(15) \qquad R_{n+1} - R_{n-1} = 2R_n$$

$$(16) \qquad R_n^2 + R_{n+1}R_{n-1} = 16P_n^2.$$

*Remark 2:* Clearly, the computing of further matrix generators can be carried out in the same fashion as above. (The reader who is patient enough may obtain as his/her reward a new Pell identity.) In the next section, we concentrate our efforts on establishing the classical Binet-type formulas mentioned in §1. To this end, we will require not only the eigenvalues but the eigenvectors of two of our matrix generators.

## SECTION 4

In (4.1), set $y = 0$, $u \neq 0$, but, for the time being, $u$ otherwise arbitrary. From §1, we know that

$$\Psi_{0u} = \begin{vmatrix} \psi & 0 \\ u & \psi' \end{vmatrix},$$

$$\Psi_{0u}^n = \begin{vmatrix} \psi^n & 0 \\ P_n u & \psi'^n \end{vmatrix}.$$

An eigenvector corresponding to the eigenvalue $\psi$ is computed to be

$$\begin{vmatrix} \dfrac{2\sqrt{2}}{u} \\ 1 \end{vmatrix};$$

while an eigenvector corresponding to $\psi'$ is $\begin{vmatrix} 0 \\ 1 \end{vmatrix}$. Now take $u = \sqrt{2}$, set $S = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}$, and simply denote $\Psi_{0\sqrt{2}}$ by $\Psi_{\sqrt{2}}$. By similarity, $\Psi_{\sqrt{2}} = S\Psi_0 S^{-1}$, which implies that $\Psi^n_{\sqrt{2}} = S\Psi^n_0 S^{-1}$, and finally that

$$\Psi^n_{\sqrt{2}} S = S\Psi^n_0. \tag{17}$$

Writing out line (17) gives

$$\begin{vmatrix} \psi^n & 0 \\ P_n\sqrt{2} & \psi'^n \end{vmatrix} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \psi^n & 0 \\ 0 & \psi'^n \end{vmatrix}. \tag{18}$$

Multiplying out in (18), we have

$$\begin{vmatrix} 2\psi^n & 0 \\ P_n 2\sqrt{2} + \psi'^n & \psi'^n \end{vmatrix} = \begin{vmatrix} 2\psi^n & 0 \\ \psi^n & \psi'^n \end{vmatrix},$$

which implies that $P_n 2\sqrt{2} + \psi'^n = \psi^n$; or, recalling that $\psi - \psi' = 2\sqrt{2}$, we have

$$P_n = \frac{\psi^n - \psi'^n}{\psi - \psi'}, \tag{19}$$

the classical Binet-type formula.

To obtain the last of the Binet-type formulas, viz., $\psi^n = \frac{1}{2}(R_n + P_n\sqrt{8})$, we use the matrix

$$N = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}.$$

A pair of eigenvectors corresponding to $\psi$, $\psi'$ are computed to be $\begin{vmatrix} \sqrt{2} \\ 1 \end{vmatrix}$, $\begin{vmatrix} -\sqrt{2} \\ 1 \end{vmatrix}$. Setting $T = \begin{vmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{vmatrix}$, and proceeding as above, we have that

$$N^n T = T\Psi^n_0;$$

i.e., that

$$\begin{vmatrix} \frac{1}{2}R_n & 2P_n \\ P_n & \frac{1}{2}R_n \end{vmatrix} \begin{vmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \psi^n & 0 \\ 0 & \psi'^n \end{vmatrix}.$$

Multiplying out gives

$$\begin{vmatrix} \frac{\sqrt{2}}{2}R_n + 2P_n & \frac{-\sqrt{2}}{2}R_n + 2P_n \\ \sqrt{2}P_n + \frac{1}{2}R_n & -\sqrt{2}P_n + \frac{1}{2}R_n \end{vmatrix} = \begin{vmatrix} \sqrt{2}\psi^n & -\sqrt{2}\psi'^n \\ \psi^n & \psi'^n \end{vmatrix}$$

which implies that

$$\psi^n = \sqrt{2}P_n + \frac{1}{2}R_n = \frac{1}{2}(\sqrt{8}P_n + R_n).$$

## REFERENCES


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# TWO THEOREMS CONCERNING HEXAGONAL NUMBERS


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Hexagonal numbers are the subset of polygonal numbers which can be expressed as $H_n = 2n^2 - n$, where $n = 1, 2, 3, \ldots$ . Geometrically hexagonal numbers can be represented as shown in Figure 1.

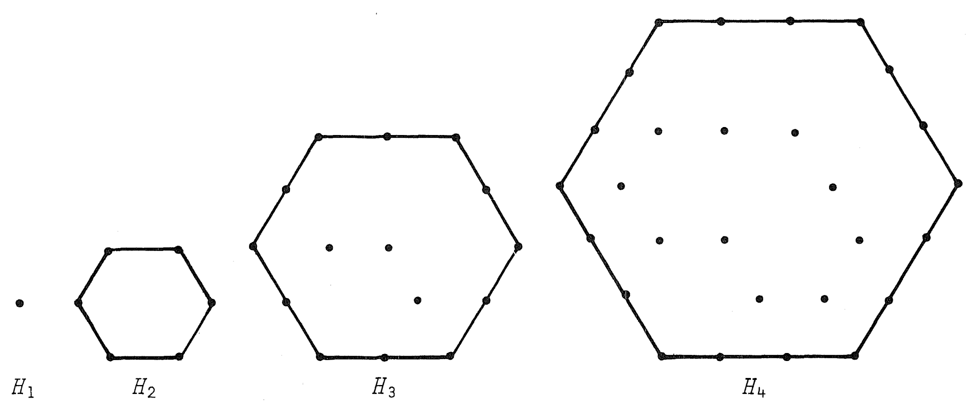


Figure 1

THE FIRST FOUR HEXAGONAL NUMBERS

Previous work by Sierpinski [1] has shown that there are an infinite number of triangular numbers which can be expressed as the sum and difference