GREATEST COMMON DIVISORS OF SUMS AND DIFFERENCES OF FIBONACCI, LUCAS, AND CHEBYSHEV POLYNOMIALS

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It is well known that the Fibonacci polynomials $F_n(x)$, the Lucas polynomials $L_n(x)$, and the Chebyshev polynomials of both kinds satisfy many "trigonometric" identities. For example, the identity

$$F_{2m}(x) + F_{2n}(x) = F_{m+n}(x)L_{|m-n|}(x) \text{ for even } m + n$$

is analogous to the trigonometric identity

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$$

Just below, we list eight well-known identities in the form which naturally results from direct proofs using the usual four identities for sums and differences of hyperbolic sines and cosines, together with certain identities in Hoggatt and Bicknell [4]:

$$F_{2n}(x) = \frac{\sinh 2n\theta}{\cosh \theta} \qquad F_{2n+1}(x) = \frac{\cosh (2n+1)\theta}{\cosh \theta}$$

$$L_{2n}(x) = 2 \cosh 2n\theta \qquad L_{2n+1}(x) = 2 \sinh (2n+1)\theta,$$

where $x = 2 \sinh \theta$. Writing simply F_n and L_n for $F_n(x)$ and $L_n(x)$ and assuming $m \geq n > 0$, the eight identities are as follows:

$$F_{2m} + F_{2n} = \begin{cases} F_{m+n}L_{m-n} & \text{if } m + n \text{ is even} \\ F_{m-n}L_{m+n} & \text{if } m + n \text{ is odd} \end{cases} \tag{1}$$

$$F_{2m} - F_{2n} = \begin{cases} F_{m-n}L_{m+n} & \text{if } m + n \text{ is even} \\ F_{m+n}L_{m-n} & \text{if } m + n \text{ is odd} \end{cases} \tag{2}$$

$$F_{2m+1} + F_{2n+1} = \begin{cases} F_{m+n+1}L_{m-n} & \text{if } m + n \text{ is even} \\ F_{m-n}L_{m+n+1} & \text{if } m + n \text{ is odd} \end{cases} \tag{3}$$

$$F_{2m+1} - F_{2n+1} = \begin{cases} F_{m-n}L_{m+n+1} & \text{if } m + n \text{ is even} \\ F_{m+n+1}L_{m-n} & \text{if } m + n \text{ is odd} \end{cases} \tag{4}$$

(5) $$L_{2m} + L_{2n} = \begin{cases} L_{m+n}L_{m-n} & \text{if } m+n \text{ is even} \\ (x^2+4)F_{m+n}F_{m-n} & \text{if } m+n \text{ is odd} \end{cases}$$

(6) $$L_{2m} - L_{2n} = \begin{cases} (x^2+4)F_{m+n}F_{m-n} & \text{if } m+n \text{ is even} \\ L_{m+n}L_{m-n} & \text{if } m+n \text{ is odd} \end{cases}$$

(7) $$L_{2m+1} + L_{2n+1} = \begin{cases} L_{m-n}L_{m+n+1} & \text{if } m+n \text{ is even} \\ (x^2+4)F_{m+n+1}F_{m-n} & \text{if } m+n \text{ is odd} \end{cases}$$

(8) $$L_{2m+1} - L_{2n+1} = \begin{cases} (x^2+4)F_{m+n+1}F_{m-n} & \text{if } m+n \text{ is even} \\ L_{m-n}L_{m+n+1} & \text{if } m+n \text{ is odd} \end{cases}$$

These identities are derived in [2] in a manner much less directly dependent on hyperbolic or trigonometric identities. See especially identities (72)-(79) in [2], which generalize considerably the present identities. An intermediate level of generalization is at the level of the generalized Fibonacci polynomials $F_n = F_n(x,z)$ and the generalized Lucas polynomials $L_n = L_n(x,z)$. For example, (5) becomes

$$L_{2m} + L_{2n} = (x^2+4z)F_{m+n}F_{m-n} \quad \text{if } m+n \text{ is odd.}$$

Let us recall the substitutions which link the F_n's and L_n's with Chebyshev polynomials $T_n(x)$ of the first kind and $U_n(x)$ of the second kind:

$$T_n(x) = \frac{1}{2}L_n(2x,-1), \quad n = 0, 1, \ldots$$

$$U_n(x) = F_{n+1}(2x,-1), \quad n = 0, 1, \ldots .$$

Clearly, our discussions involving F_n's and L_n's carry over immediately to T_n's and U_n's; bearing this in mind, we make no further mention of Chebyshev polynomials in this paper.

Identities (1)-(8) show that greatest common divisors for certain sums and differences of the various polynomials can be found in terms of the irreducible divisors of individual generalized Fibonacci polynomials and generalized Lucas polynomials. In [7], we showed these divisors to be the generalized Fibonacci-cyclotomic polynomials $\mathcal{F}_n(x,z)$. The interested reader should consult [7] for a definition of these polynomials. Theorems 6 and 10 in [7] may be restated for $n \geq 1$ as follows:

(I) $$F_n(x,z) = \prod_{d|n} \mathcal{F}_d(x,z)$$

(II) $$L_n(x,z) = \prod_{d|q} \mathcal{F}_{2^{t+1}d}(x,z), \text{ where } n = 2^t q,\ q \text{ odd},\ t \geq 0.$$

The (ordinary) Fibonacci and Lucas polynomials are given by $F_n(x) = F_n(x,1)$ and $L_n(x) = L_n(x,1)$, and their factorizations as products of the irreducible polynomials $\mathcal{F}(x) = \mathcal{F}(x,1)$ are given by (I) and (II). With these factorizations, we are able to prove the following theorem.

Theorem 1: For any nonnegative integers a, b, c, d, the greatest common divisor of L_aF_b and L_cF_d is given by

$$(L_aF_b, L_cF_d) = F_{(b,d)}\ \frac{F_{(b,2c)} \cdot F_{(b,c,d)} \cdot F_{(2a,d)} \cdot F_{(a,b,d)} \cdot F_{(2a,2c)} \cdot F_{(a,c)}}{F_{(b,c)} \cdot F_{(b,2c,d)} \cdot F_{(a,d)} \cdot F_{(2a,b,d)} \cdot F_{(2a,c)} \cdot F_{(a,2c)}} \quad \text{times}$$

$$\frac{F_{(2a,b,c)} \cdot F_{(a,b,2c)} \cdot F_{(2a,c,d)} \cdot F_{(a,2c,d)} \left[F_{(2a,b,2c,d)} \cdot F_{(a,b,c,d)}\right]^2}{F_{(2a,b,2c)} \cdot F_{(a,b,c)} \cdot F_{(2a,2c,d)} \cdot F_{(a,c,d)} \left[F_{(2a,b,c,d)} \cdot F_{(a,b,2c,d)}\right]^2}.$$

Proof: Write $a = 2^s\alpha$, α odd, and $c = 2^t\gamma$, γ odd. Let

$$A = \{\delta : \delta = 2^{s+1}q \text{ for some } q \text{ satisfying } q|\alpha\}$$

$$C = \{\delta : \delta = 2^{t+1}q \text{ for some } q \text{ satisfying } q|\gamma\}$$

$$B = \{\delta : \delta|b\} \quad \text{and} \quad D = \{\delta : \delta|d\}.$$

In terms of these sets, let

$$S_1 = B \cap D$$

$$S_2 = B \cap C - B \cap C \cap D$$

$$S_3 = A \cap D - A \cap B \cap D$$

$$S_4 = A \cap C - A \cap S_2 - C \cap S_3.$$

Then,

$$(L_aF_b, L_cF_d) = \left(\prod_{\delta \in A}\mathcal{F}_\delta \prod_{\delta \in B}\mathcal{F}_\delta, \prod_{\delta \in C}\mathcal{F}_\delta \prod_{\delta \in D}\mathcal{F}_\delta\right) = \prod_{i=1}^{4} \prod_{\delta \in S_i}\mathcal{F}_\delta.$$

One may now readily verify that $\prod_{\delta \in S_1}\mathcal{F}_\delta = F_{(b,d)}$,

$$\prod_{\delta \in S_2}\mathcal{F}_\delta = \frac{F_{(b,2c)}}{F_{(b,c)}} \div \frac{F_{(b,2c,d)}}{F_{(b,c,d)}} \quad \text{and} \quad \prod_{\delta \in S_3}\mathcal{F}_\delta = \frac{F_{(2a,d)}}{F_{(a,d)}} \div \frac{F_{(2a,b,d)}}{F_{(a,b,d)}}.$$

For the product involving S_4, we have

$$\prod_{\delta \in A \cap C}\mathcal{F}_\delta = \frac{F_{(2a,2c)} \cdot F_{(a,c)}}{F_{(2a,c)} \cdot F_{(a,2c)}},$$

$$\prod_{\delta \in A \cap S_2}\mathcal{F}_\delta = \frac{F_{(2a,b,2c)} \cdot F_{(a,b,c)}}{F_{(2a,b,c)} \cdot F_{(a,b,2c)}} \div \frac{F_{(2a,b,c,d)} \cdot F_{(a,b,2c,d)}}{F_{(2a,b,2c,d)} \cdot F_{(a,b,c,d)}}, \text{ and}$$

$$\prod_{\delta \in A \cap S_3}\mathcal{F}_\delta = \frac{F_{(2a,d,2c)} \cdot F_{(a,d,c)}}{F_{(2a,d,c)} \cdot F_{(a,d,2c)}} \div \frac{F_{(2a,b,c,d)} \cdot F_{(a,b,2c,d)}}{F_{(2a,b,2c,d)} \cdot F_{(a,b,c,d)}}.$$

Now using

$$\prod_{\delta \in S_4} \mathcal{F}_\delta = \prod_{\delta \in A \cap C} \mathcal{F}_\delta \div \prod_{\delta \in A \cap S_2} \mathcal{F}_\delta \div \prod_{\delta \in A \cap S_3} \mathcal{F}_\delta ,$$

the desired formula is easily put together.

Corollary: $(L_a, L_c) = \dfrac{F_{(2a,2c)} \cdot F_{(a,c)}}{F_{(2a,c)} \cdot F_{(a,2c)}}$.

It is easy to obtain formulas for $(F_a F_b, F_c F_d)$ and $(L_a L_b, L_c L_d)$ using the method of proof of Theorem 1. The Lucas-formula has the same form as that in Theorem 1, but even more factors. The Fibonacci-formula too has this form, but few enough factors that we choose to include it here:

$$(F_a F_b, F_c F_d) = F_{(b,d)} \frac{F_{(b,c)} \cdot F_{(a,d)} \cdot F_{(a,c)} \cdot F^2_{(a,b,c,d)}}{F_{(b,c,d)} \cdot F_{(a,b,d)} \cdot F_{(a,b,c)} \cdot F_{(a,c,d)}} .$$

Returning now to sums and differences of polynomials, we find from identities (1) and (3), for example, that

(1′) $\quad F_{4k+n} + F_n = L_{2k} F_{2k+n}$ for any nonnegative integers k and n.

Thus, Theorem 1 enables us to write out the greatest common divisor of any two terms of the sequence

$$F_4,\ F_5 + F_1,\ F_6 + F_2,\ F_7 + F_3,\ \ldots$$

or of the sequence

$$F_1 + 1,\ F_5 + 1,\ F_9 + 1,\ F_{13} + 1,\ \ldots .$$

With the help of (3′) below, we can refine the latter sequence to

$$F_1 + 1,\ F_3 + 1,\ F_5 + 1,\ F_7 + 1,\ \ldots$$

and still find greatest common divisors. (But what about the sequence $\{F_n + 1\}$ for *all* positive integers n?)

Following is a list of double-sequence identities like (1′). These are easily obtained from identities (1)-(8).

(1′) $\quad F_{4k+n} + F_n = L_{2k} F_{2k+n}$

(2′) $\quad F_{4k+n} - F_n = F_{2k} L_{2k+n}$

(3′) $\quad F_{4k+n+2} + F_n = L_{2k+n+1} F_{2k+1}$

(4′) $\quad F_{4k+n+2} - F_n = F_{2k+n+1} L_{2k+1}$

(5′) $$L_{4k+n} + L_n = L_{2k}L_{2k+n}$$

(6′) $$L_{4k+n} - L_n = (x^2 + 4)F_{2k}F_{2k+n}$$

(7′) $$L_{4k+n+2} + L_n = (x^2 + 4)F_{2k+1}F_{2k+n+1}$$

(8′) $$L_{4k+n+2} - L_n = L_{2k+1}L_{2k+n+1}.$$

We note that the divisibility properties of some of these sequences are much the same as those of the sequence of Fibonacci polynomials [namely, $(F_m, F_n) = F_{(m,n)}$ with F_p irreducible over the integers whenever p is a prime] or the sequence of Lucas polynomials. For example, the sequence s_0, s_1, s_2, ..., given by

$$0,\ L_2 + 2,\ L_4 - 2,\ L_6 + 2,\ L_8 - 2,\ \ldots,$$

has $(s_m, s_n) = (x^2 + 4)F^2_{(m,n)}$ for all positive integers m and n.

One might expect Theorem 1 to apply to sequences other than (1′)-(8′) in the manner just exemplified. A good selection of forty identities, some admitting applications of Theorem 1, is found in [3], pp. 52-59.

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