Multiplying out gives

$$\begin{vmatrix} \frac{\sqrt{2}}{2}R_n + 2P_n & \frac{-\sqrt{2}}{2}R_n + 2P_n \\ \sqrt{2}P_n + \frac{1}{2}R_n & -\sqrt{2}P_n + \frac{1}{2}R_n \end{vmatrix} = \begin{vmatrix} \sqrt{2}\psi^n & -\sqrt{2}\psi'^n \\ \psi^n & \psi'^n \end{vmatrix}$$

which implies that

$$\psi^n = \sqrt{2}P_n + \frac{1}{2}R_n = \frac{1}{2}(\sqrt{8}P_n + R_n).$$

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TWO THEOREMS CONCERNING HEXAGONAL NUMBERS

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Hexagonal numbers are the subset of polygonal numbers which can be expressed as $H_n = 2n^2 - n$, where $n = 1, 2, 3, \ldots$. Geometrically hexagonal numbers can be represented as shown in Figure 1.

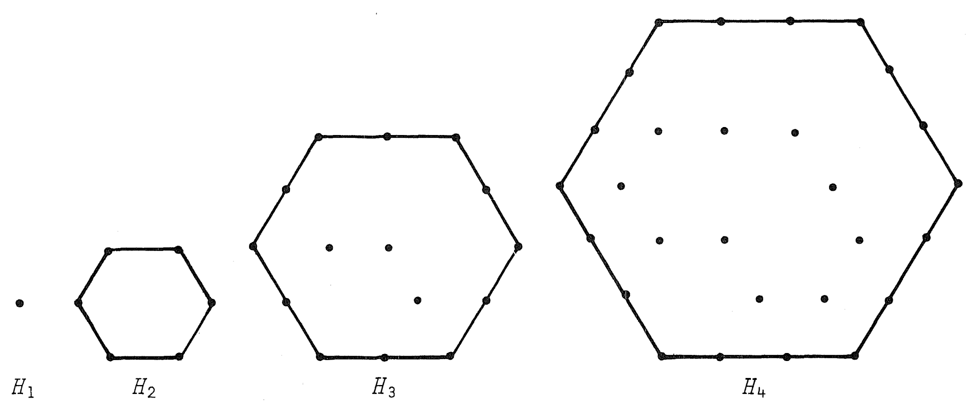

Figure 1

THE FIRST FOUR HEXAGONAL NUMBERS

Previous work by Sierpinski [1] has shown that there are an infinite number of triangular numbers which can be expressed as the sum and difference of triangular numbers, while Hansen [2] has proved a similar result for pentagonal numbers. This paper will present a proof that there are an infinite number of hexagonal numbers which can be expressed as the sum and difference of hexagonal numbers.

A table of hexagonal numbers is shown in Table 1.

Table 1

THE FIRST 100 HEXAGONAL NUMBERS

1	6	15	28	45	66	91	120	153	190
231	276	325	378	435	496	561	630	703	780
861	946	1035	1128	1225	1326	1431	1540	1653	1770
1891	2016	2145	2278	2415	2556	2701	2850	3003	3160
3321	3486	3655	3828	4005	4186	4371	4560	4753	4950
5151	5356	5565	5778	5995	6216	6441	6670	6903	7140
7381	7626	7875	8128	8385	8646	8911	9180	9453	9730
10011	10296	10585	10878	11175	11476	11781	12090	12403	12720
13041	13366	31695	14028	14365	14706	15051	15400	15753	16110
16471	16836	17205	17578	17955	18336	18721	19110	19503	19900

It is noted that

$$\begin{aligned} H_n - H_{n-1} &= [2n^2 - n] - [2(n - 1)^2 - (n - 1)] \\ &= 2n^2 - n - 2n^2 + 5n - 3 \\ &= 4n - 3. \end{aligned}$$

We observe that

(a) $H_{12} = H_5 + H_{11}$

(b) $H_{39} = H_9 + H_{38}$

(c) $H_{82} = H_{13} + H_{81}$

In each instance $H_m = H_{4n+1} + H_{m-1}$ for $n = 1, 2, 3, \ldots$. We note that

$$\begin{aligned} H_{4n+1} &= 2(4n + 1)^2 - (4n + 1) \\ &= 32n^2 + 12n + 1. \end{aligned}$$

From the previous work, it is clear that

$$H_j - H_{j-1} = 4j - 3 = 32n^2 + 12n + 1, \text{ for some } n.$$

Solving for j, we find that

$$j = 8n^2 + 3n + 1,$$

which is an integer. These results yield the following theorem.

Theorem 1: $H_{8n^2+3n+1} = H_{4n+1} + H_{8n^2+3n}$ for any integer $n \geq 1$.

For $n = 1, 2, 3, \ldots$, we have directly from Theorem 1 that

$$H_{8(4n)^2+3(4n)+1} = H_{4(4n)+1} + H_{8(4n)^2+3(4n)}$$

or

(1) $$H_{128n^2+12n+1} = H_{16n+1} + H_{128n^2+12n}.$$

Now consider $H_{128n^2+12n+1} = H_k - H_{k-1} = 4k - 3$. Then,

$$\begin{aligned} H_{128n^2+12n+1} &= 2(128n^2 + 12n + 1)^2 - (128n^2 + 12n + 1) \\ &= 32768n^4 + 6144n^3 + 672n^2 + 36n + 1 = 4k - 3. \end{aligned}$$

Solving for k, we find

$$k = 8192n^4 + 1536n^3 + 168n^2 + 9n + 1,$$

which is an integer. We now have

$$\begin{aligned} H_{128n^2+12n+1} &= H_{8192n^4+1536n^3+168n^2+9n+1} \\ &\quad - H_{8192n^4+1536n^3+168n^2+9n}. \end{aligned} \tag{2}$$

Combining equations (1) and (2), we have the following theorem.

Theorem 2: For any integer $n \geq 1$,

$$\begin{aligned} H_{128n^2+12n+1} &= H_{16n+1} + H_{128n^2+12n} \\ &= H_{8192n^4+1536n^3+168n^2+9n} \\ &\quad - H_{8192n^4+1536n^3+168n^2+9n} \end{aligned}$$

For $n = 1, 2$, we have

$$\begin{aligned} H_{141} &= H_{17} + H_{140} \\ &= H_{9906} - H_{9905} \end{aligned}$$

or

$$\begin{aligned} 39{,}621 &= 561 + 39{,}061 \\ &= 196{,}247{,}766 - 196{,}208{,}145 \end{aligned}$$

and

$$\begin{aligned} H_{537} &= H_{33} + H_{536} \\ &= H_{144051} - H_{144050} \end{aligned}$$

or

$$\begin{aligned} 576{,}201 &= 2145 + 574{,}056 \\ &= 41{,}501{,}237{,}151 - 41{,}500{,}660{,}950. \end{aligned}$$

CONCLUSION

Theorem 2 establishes that there are an infinite number of hexagonal numbers which can be expressed as the sum and difference of hexagonal numbers. This result, along with the results of Sierpinski and Hansen, suggests that for any fixed polygonal number there are an infinite number of polygonal numbers which can be expressed as the sum and difference of similar polygonal numbers. A proof of this fact, though, is unknown to the author.

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