SOME CONGRUENCES INVOLVING GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

Throughout this paper, let $\{H_n\}$ be the generalized Fibonacci sequence defined by

(1) $$H_0 = q, \quad H_1 = p, \quad H_{n+1} = H_n + H_{n-1},$$

and let $\{V_n\}$ be the generalized Lucas sequence defined by

(2) $$V_n = H_{n+1} + H_{n-1}.$$

If $q = 0$ and $p = 1$, $\{H_n\}$ becomes $\{F_n\}$, the Fibonacci sequence, and $\{V_n\}$ becomes $\{L_n\}$, the Lucas sequence. We use the recursion formula to extend to negative subscripts the definition of each of these sequences.

Our purpose here is to examine several consequences of the identities

(3) $$H_{n+r} + (-1)^r H_{n-r} = L_r H_n$$

and

(4) $$H_{n+r} - (-1)^r H_{n-r} = F_r V_n,$$

both of which were given several years ago in my master's thesis [12]. Identity (3) has been reported several times: by Tagiuri [5], by Horadam [8], and more recently by King and Hosford [10]. However, identity (4) seems to have escaped attention.

We will first establish identities (3) and (4), and then show how they can be used to solve several problems which have appeared in these pages in the past. We close with a generalization of the identities.

2. PROOF OF THE IDENTITIES

The Binet formulas

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, easily generalize to

$$H_n = (A\alpha^n - B\beta^n)/\sqrt{5} \quad \text{and} \quad V_n = A\alpha^n + B\beta^n,$$

where $A = p - q\beta$ and $B = p - q\alpha$. Any of these formulas may be obtained easily by standard finite difference techniques, or may be verified by induction.

Since $\alpha\beta = -1$, we have

$$\begin{aligned} H_{n+r} + (-1)^r H_{n-r} &= \left\{A\alpha^{n+r} - B\beta^{n+r} + \alpha^r\beta^r A\alpha^{n-r} - \alpha^r\beta^r B\beta^{n-r}\right\}/\sqrt{5} \\ &= \left\{A\alpha^n\alpha^r + A\alpha^n\beta^r - B\beta^n\alpha^r - B\beta^n\beta^r\right\}/\sqrt{5} \\ &= \left\{\alpha^r + \beta^r\right\}\cdot\left\{A\alpha^n - B\beta^n\right\}/\sqrt{5} \\ &= L_r H_n. \end{aligned}$$

Therefore, (3) is established.

Similarly,

$$\begin{aligned} H_{n+r} - (-1)^r H_{n-r} &= \left\{A\alpha^{n+r} - B\beta^{n+r} - \alpha^r\beta^r A\alpha^{n-r} + \alpha^r\beta^r B\beta^{n-r}\right\}/\sqrt{5} \\ &= \left\{A\alpha^n\alpha^r + B\beta^n\alpha^r - A\beta^n\alpha^r - B\beta^n\beta^r\right\}/\sqrt{5} \\ &= \left\{A\alpha^n + B\beta^n\right\}\cdot\left\{\alpha^r - \beta^r\right\}/\sqrt{5} \\ &= F_r V_n , \end{aligned}$$

so (4) is also verified.

3. CONSEQUENCES OF THE IDENTITIES

It is sometimes more convenient to rewrite identities (3) and (4) as

$$H_{k+2h} - H_k = \begin{cases} L_h H_{k+h} & (h \text{ odd}) \\ F_h V_{k+h} & (h \text{ even}) \end{cases} \tag{5}$$

and

$$H_{k+2h} + H_k = \begin{cases} F_h V_{k+h} & (h \text{ odd}) \\ L_h H_{k+h} & (h \text{ even}). \end{cases} \tag{6}$$

In the discussion which follows, it is helpful to remember that:

i. If $H_n = F_n$, then $V_n = L_n$.

ii. If $H_n = L_n$, then $V_n = 5F_n$.

iii. For all k, F_n divides F_{nk}.

iv. If k is odd, then L_n divides L_{nk}.

By (5), we have

$$H_{n+24} - H_n = F_{12} V_{n+12} = 144 V_{n+12}.$$

Therefore, with $H_n = F_n$,

$$F_{n+24} \equiv F_n \pmod 9,$$

as asserted in problem B-3 [9].

Direct application of (5) yields

$$H_{n+4m+2} - H_n = L_{2m+1} H_{n+2m+1}$$

so that

$$F_{n+4m+2} - F_n = L_{2m+1} F_{n+2m+1},$$

as claimed in problem B-17 [13].

Since $L_0 = 2$, identity (4) gives us

$$\begin{aligned} L_{2k} - 2(-1)^k &= L_{k+k} - (-1)^k L_{k-k} \\ &= F_k(5F_k) = 5F_k^2. \end{aligned}$$

Therefore, $L_{2k} \equiv 2(-1)^k \pmod 5$, which was the claim of problem B-88 [14].

If k is odd, then (5) tells us that

$$H_{nk+2k} - H_{nk} = L_k H_{nk+k},$$

so

$$F_{(n+2)k} \equiv F_{nk} \pmod{L_k} \quad (k \text{ odd})$$

as asserted in problem B-270 [6].

By (6) we have

$$\begin{aligned} F_{8n-4} + F_{8n} + F_{8n+4} &= F_n + F_{8n+4} + F_{8n-4} \\ &= F_{8n} + L_4 F_{8n} = (1 + 7)F_{8n} = 8F_{8n}. \end{aligned}$$

Since $21 = F_8$ divides F_{8n}, it follows that

$$F_{8n-4} + F_{8n} + F_{8n+4} \equiv 0 \pmod{168}$$

as claimed in problem B-203 [7].

In problem B-31 [11], Lind asserted that if n is even, then the sum of $2n$ consecutive Fibonacci numbers is divisible by F_n. We will establish a stronger result. Horadam [8] showed that

$$H_1 + H_2 + \cdots + H_{2n} = H_{2n+2} - H_2.$$

If n is even, then by (5) we have

$$H_1 + H_2 + \cdots + H_{2n} = H_{2n+2} - H_2 = F_n V_{n+2},$$

which is clearly divisible by F_n. Because the sum of $2n$ consecutive generalized Fibonacci numbers is the sum of the first $2n$ terms of another generalized Fibonacci sequence (obtained by a simple shift), Lind's result holds for generalized Fibonacci numbers. In addition, we may similarly conclude from (5) that if n is odd, the sum of $2n$ consecutive generalized Fibonacci numbers is divisible by L_n.

By (5),

$$\begin{aligned} H_{2n(2k+1)} - H_{2n} &= H_{2n+4nk} - H_{2n} \\ &= F_{2nk} V_{2n+2nk} = F_{2nk} V_{2n(k+1)}. \end{aligned}$$

Therefore (with $H_n = L_n$ and $V_n = 5F_n$)

$$L_{2n(2k+1)} - L_{2n} = 5F_{2nk} F_{2n(k+1)},$$

so not only is it true that

$$L_{2n(2k+1)} \equiv L_{2n} \pmod{F_{2n}},$$

as asserted in problem B-277 [1], but indeed

$$L_{2n(2k+1)} \equiv L_{2n} \pmod{F_{2n}^2}$$

since F_{2n} divides both F_{2nk} and $F_{2n(k+1)}$.

In a similar fashion,

$$\begin{aligned} H_{(2n+1)(4k+1)} - H_{2n+1} &= H_{2n+1+4k(2n+1)} - H_{2n+1} \\ &= F_{2k(2n+1)} V_{2n+1+2k(2n+1)} \\ &= F_{2k(2n+1)} V_{(2k+1)(2n+1)} \end{aligned}$$

so that

$$L_{(2n+1)(4k+1)} - L_{2n+1} = 5F_{2k(2n+1)} F_{(2k+1)(2n+1)}.$$

Therefore,

$$L_{(2n+1)(4k+1)} \equiv L_{2n+1} \pmod{F_{2n+1}^2}$$

and in particular

$$L_{(2n+1)(4k+1)} \equiv L_{2n+1} \pmod{F_{2n+1}}$$

as claimed in problem B-278 [2].

Also,

$$\begin{aligned} H_{2n(4k+1)} - H_{2n} &= H_{2n+8nk} - H_{2n} \\ &= F_{4nk} V_{2n+4nk} \\ &= F_{4nk} V_{2n(2k+1)} . \end{aligned}$$

Therefore,

$$F_{2n(4k+1)} - F_{2n} = F_{4nk} L_{2n(2k+1)} ,$$

so

$$F_{2n(4k+1)} \equiv F_{2n} \pmod{L_{2n(2k+1)}}.$$

Since L_{2n} divides $L_{2n(2k+1)}$, we have

$$F_{2n(4k+1)} \equiv F_{2n} \pmod{L_{2n}},$$

which establishes problem B-288 [3].

Now let us consider

$$H_{(2n+1)(2k+1)} - H_{2n+1} = H_{2n+1+2k(2n+1)} - H_{2n+1}.$$

By (5) we have

$$\begin{cases} L_{k(2n+1)} H_{(k+1)(2n+1)} & \text{if } k \text{ is odd,} \\ F_{k(2n+1)} V_{(k+1)(2n+1)} & \text{if } k \text{ is even.} \end{cases}$$

Therefore

$$F_{(2n+1)(2k+1)} - F_{2n+1} = \begin{cases} L_{k(2n+1)} F_{(k+1)(2n+1)} & \text{if } k \text{ is odd} \\ F_{k(2n+1)} L_{(k+1)(2n+1)} & \text{if } k \text{ is even.} \end{cases}$$

If k is odd, $L_{k(2n+1)}$ is divisible by L_{2n+1}; if k is even, then $k+1$ is odd, so L_{2n+1} divides $L_{(k+1)(2n+1)}$. Hence, in any case,

$$F_{(2n+1)(2k+1)} \equiv F_{2n+1} \pmod{L_{2n+1}},$$

which was the claim in problem B-289 [4].

Finally, we note that adding (3) and (4) yields

$$F_{n+r} = (L_r F_n + F_r L_n)/2$$

if $H_n = F_n$ (and $V_n = L_n$), and

$$L_{n+r} = (L_r L_n + 5F_r F_n)/2$$

if $H_n = L_n$ (and $V_n = 5F_n$). Subtraction of the same two identities gives us

$$F_{n-r} = (-1)^r (L_r F_n - F_r L_n)/2$$

and

$$L_{n-r} = (-1)^r (L_r L_n - 5F_r F_n)/2.$$

These results appear to be new.

4. GENERALIZATION OF THE IDENTITIES

Let $\{u_n\}$ be the generalized second order recurring sequence defined by

$$u_0 = q, \quad u_1 = p, \quad u_{n+1} = gu_n + hu_{n-1},$$

where $g^2 + 4h \neq 0$ (to avoid having repeated roots of the associated finite difference equation). Define $\{v_n\}$ by

$$v_n = u_{n+1} + hu_{n-1},$$

let $\{s_n\}$ be defined by

$$s_0 = 0, \quad s_1 = 1, \quad s_{n+1} = gs_n + hs_{n-1},$$

and let $\{t_n\}$ be defined by

$$t_n = s_{n+1} + hs_{n-1}.$$

Extend each sequence to negative subscripts by means of the recurrence relation.

Then if

$$\alpha = \left(g + \sqrt{g^2 + 4h}\right)/2 \quad \text{and} \quad \beta = \left(g - \sqrt{g^2 + 4h}\right)/2,$$

the Binet-like identities are easy to prove:

$$s_n = (\alpha^n - \beta^n)/(\alpha - \beta)$$

$$t_n = \alpha^n + \beta^n$$

$$u_n = (A\alpha^n - B\beta^n)/(\alpha - \beta)$$

$$v_n = A\alpha^n + B\beta^n,$$

where $A = p - q\beta$ and $B = p - q\alpha$.

Then it is a simple matter to establish that

$$u_{n+r} + (-h)^r u_{n-r} = t_r u_n$$

and

$$u_{n+r} - (-h)^r u_{n-r} = s_r v_n.$$

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