and since $\left(\frac{-2}{5}\right) = \left(\frac{2}{5}\right) = -1$, (13) is impossible.

(q) (13) is impossible if $n \equiv 7 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv u_7 \pmod{\eta_5} \\ &\equiv 37 \pmod{11} \\ &\equiv 26 \pmod{11}. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 13 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{13}{11}\right) = -1$, (13) is impossible.

(r) (13) is impossible if $n \equiv 9 \pmod{10}$, for, using (11) we find that

$$\begin{aligned} u_n &\equiv u_9 \pmod{\eta_5} \\ &\equiv 97 \pmod{11} \\ &\equiv 86 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 43 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{43}{11}\right) = -1$, (13) is impossible.

Hence, none of the pseudo-Fibonacci numbers are of the form $2S^2$, where S is an integer.

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INFINITE SERIES WITH FIBONACCI AND LUCAS POLYNOMIALS

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In [7], D. A. Millin poses the problem of showing that

$$\sum_{n=0}^{\infty} F_{2^n}^{-1} = \frac{7 - \sqrt{5}}{2} \tag{1}$$

where F_k is the kth Fibonacci number. A proof of (1) by I. J. Good is given in [5], while in [3], Hoggatt and Bicknell demonstrate ten different methods of finding the same sum. Furthermore, the result of (1) is extended by Hoggatt and Bicknell in [4], where they show that

(2) $$\sum_{n=0}^{\infty} F_{2^n k}^{-1} = \frac{1}{F_k} + \frac{\alpha^2 + 1}{\alpha(\alpha^{2k} - 1)}.$$

The main purpose of this paper is to lift the results of (1) and (2) to the sequence of Fibonacci polynomials $\{F_k(x)\}_{k=1}^{\infty}$ defined recursively by

$$F_1(x) = 1,\ F_2(x) = x,\ F_{k+2}(x) = xF_{k+1}(x) + F_k(x),\ k \geq 1.$$

Furthermore, we will examine several infinite series containing products of Fibonacci and Lucas polynomials where the Lucas polynomials are defined by

$$L_k(x) = F_{k+1}(x) + F_{k-1}(x).$$

If we let $\alpha(x) = (x + \sqrt{x^2 + 4})/2$ and $\beta(x) = (x - \sqrt{x^2 + 4})/2$, then it is a well-known fact that

(3) $$F_k(x) = [\alpha^k(x) - \beta^k(x)]/[\alpha(x) - \beta(x)]$$

and

(4) $$L_k(x) = \alpha^k(x) + \beta^k(x).$$

When $x > 0$, we have $-1 < \beta(x) < 1$ and $\alpha(x) > 1$ so that $|\beta(x)/\alpha(x)| < 1$ and $\lim_{n\to\infty}[\beta(x)/\alpha(x)]^n = 0$. But, from (3), we obtain

(5) $$\frac{F_{n+1}(x)}{F_n(x)} = \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha^n(x) - \beta^n(x)} = \frac{\alpha(x) - \beta(x)}{1 - [\beta(x)\alpha^{-1}(x)]^n} + \beta(x).$$

Therefore,

(6) $$\lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \alpha(x), \text{ if } x > 0.$$

When $x < 0$, we have $0 < \alpha(x) < 1$ and $\beta(x) < -1$ so that $\beta(x)/\alpha(x) < -1$. From (5), we see that

(7) $$\lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \beta(x), \text{ if } x < 0.$$

Using (3) and (4), it is easy to show that

$$L_{n+k}(x) + L_{n-k}(x) = L_n(x)L_k(x),\ k \text{ even}$$

and

$$F_{2n}(x) = L_n(x)F_n(x).$$

Letting S_n be the nth partial sum of

$$\sum_{n=1}^{\infty} xF_{2^n k}^{-1}(x)$$

and using the two preceding equations with induction, it can be shown that

$$S_n = \frac{x}{F_{2^n k}(x)}\left[\sum_{t=1}^{2^{n-1}-1} L_{2^n k - 2kt}(x) + 1\right].$$

The definition of $L_k(x)$ together with (6) enables us to show for $x > 0$ that

$$\lim_{n\to\infty} S_n = x\sum_{t=1}^{\infty} \frac{\alpha^2(x) + 1}{\alpha^{2kt+1}(x)} = \frac{[\alpha^2(x) + 1]x}{\alpha(x)[\alpha^{2k}(x) - 1]}$$

while for $x < 0$ we use (7) to obtain

$$\lim_{n\to\infty} S_n = x\sum_{t=1}^{\infty}\frac{\beta^2(x)+1}{\beta^{2kt+1}(x)} = \frac{[\beta^2(x)+1]x}{\beta(x)[\beta^{2k}(x)-1]}$$

Hence,

$$(8)\qquad \sum_{n=0}^{\infty} xF_{2^n k}^{-1}(x) = \frac{x}{F\ (x)} + \begin{cases}\left[(\alpha^2(x)+1)x\right]\Big/\left[\alpha(x)(\alpha^{2k}(x)-1)\right], & x > 0\\ \left[(\beta^2(x)+1)x\right]\Big/\left[\beta(x)(\beta^{2k}(x)-1)\right], & x < 0\end{cases}.$$

We now examine the infinite series

$$(9)\qquad U(q,a,b,x) = \sum_{n=1}^{\infty}\frac{(-1)^{qn+a-k}F_{b-a+k}(x)F_k(x)}{F_{qn+a-k}(x)F_{qn+b}(x)},\quad q = b - a + k.$$

First observe that, by using (3) and (4), we can show

$$(10)\qquad F_{qn+a}(x)F_{qn+b}(x) - F_{qn+a-k}(x)F_{qn+b+k}(x) = (-1)^{qn+a-k}F_k(x)F_{b-a+k}(x).$$

Letting S_n be the nth partial sum of (9) and using (10), we notice that there is a telescoping effect so that

$$S_n = \frac{F_{b+k}(x)}{F_b(x)} - \frac{F_{qn+b+k}(x)}{F_{qn+b}(x)}.$$

Hence, by (6) and (7), we have

$$(11)\qquad U(q,a,b,x) = \frac{F_{b+k}(x)}{F_b(x)} - \begin{cases}\alpha^k(x), & x > 0\\ \beta^k(x), & x < 0\end{cases},$$

where $q = b - a + k$. In particular, we see that

$$(12)\qquad U(a,a,a,x) = \sum_{n=1}^{\infty}\frac{(-1)^{an}F_a^2(x)}{F_{an}(x)F_{a(n+1)}(x)} = L_a(x) - \begin{cases}\alpha^a(x), & x > 0\\ \beta^a(x), & x < 0\end{cases},$$

$$(13)\qquad U(1,1,1,x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{F_n(x)F_{n+1}(x)} = \begin{cases}\beta(x), & x > 0\\ \alpha(x), & x < 0\end{cases},$$

$$(14)\qquad U(2,2,2,x) = \sum_{n=1}^{\infty}\frac{x^2}{F_{2n}(x)F_{2(n+1)}(x)} = \begin{cases}x^2 - x\alpha(x) + 1, & x > 0\\ x^2 - x\beta(x) + 1, & x < 0\end{cases},$$

and

$$(15)\qquad U(b,1,b,x) = \sum_{n=1}^{\infty}\frac{(-1)^{bn}F_b(x)}{F_{bn}(x)F_{b(n+1)}(x)} = \frac{F_{b+1}(x)}{F_b(x)} - \begin{cases}\alpha(x), & x > 0\\ \beta(x), & x < 0\end{cases}.$$

If we combine (13) and (14) with the identity

$$L_{2n+1}(x) = L_n(x)L_{n+1}(x) + (-1)^{n+1}x$$

we obtain the very interesting result

(16) $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} L_{2n+1}(x)}{F_{2n}(x) F_{2(n+1)}(x)} = \frac{1}{x}.$$

Next, we examine the infinite series

(17) $$V(q,a,b,x) = -\sum_{n=1}^{\infty} \frac{(x^2+4)(-1)^{qn+a-k} F_k(x) F_{b-a+k}(x)}{L_{qn+a-k}(x) L_{qn+b}(x)},$$

$$q = b - a + k.$$

To do this, we first use (3) and (4) to show that

(18) $$L_{qn+a}(x) L_{qn+b}(x) - L_{qn+a-k}(x) L_{qn+b+k}(x)$$
$$= -(x^2+4)(-1)^{qn+a-k} F_k(x) F_{b-a+k}(x).$$

Letting S_n be the nth partial sum of (17) and using (18), we notice that there is a telescoping effect so that

$$S_n = \frac{L_{b+k}(x)}{L_b(x)} - \frac{L_{qn+b+k}(x)}{L_{qn+b}(x)}.$$

Using the definition of $L_m(x)$ together with (6) and (7), we obtain

(19) $$V(q,a,b,x) = \frac{L_{b+k}(x)}{L_b(x)} - \begin{cases} \alpha^k(x), & x > 0 \\ \beta^k(x), & x < 0 \end{cases}$$

where $q = b - a + k$. In particular, we note that

(20) $$V(a,a,a,x) = -\sum_{n=1}^{\infty} \frac{(x^2+4)(-1)^{an} F_a^2(x)}{L_{an}(x) L_{a(n+1)}(x)} = \frac{L_{2a}(x)}{L_a(x)} - \begin{cases} \alpha^a(x), & x > 0 \\ \beta^a(x), & x < 0 \end{cases},$$

(21) $$V(b,1,b,x) = -\sum_{n=1}^{\infty} \frac{(x^2+4)(-1)^{bn} F_b(x)}{L_{bn}(x) L_{b(n+1)}(x)} = \frac{L_{b+1}(x)}{L_b(x)} - \begin{cases} \alpha(x), & x > 0 \\ \beta(x), & x < 0 \end{cases}.$$

In conclusion, we observe that

(22) $$F_{n-1}(x) F_{n+1}(x) - F_{n+2}(x) F_{n-2}(x) = (-1)^n (x^2+1).$$

Letting S_n be the nth partial sum of

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x^2+1)}{F_{n+1}(x) F_{n+2}(x)}$$

and using (22), we see that

$$S_n = -\frac{F_{-1}(x)}{F_2(x)} + \frac{F_{n-1}(x)}{F_{n+2}(x)} = -\frac{1}{x} + \frac{F_{n-1}(x)}{F_{n+2}(x)}$$

so that

(23) $$\sum_{n=1}^{\infty} \frac{(-1)^n (x^2+1)}{F_{n+1}(x) F_{n+2}(x)} = -\frac{1}{x} - \begin{cases} \beta^3(x), & x > 0 \\ \alpha^3(x), & x < 0 \end{cases}.$$

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A NOTE ON 3-2 TREES*

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ABSTRACT

Under the assumption that all of the 3-2 trees of height h are equally probable, it is shown that in a 3-2 tree of height h the expected number of keys is $(.72162)3^h$ and the expected number of internal nodes is $(.48061)3^h$.

INTRODUCTION

One approach to the organization of large files is the use of "balanced" trees (see Section 6.2.3 of [3]). In particular, one such class of trees, suggested by J. E. Hopcroft (unpublished), is known as 3-2 trees. A 3-2 tree is a tree in which each internal node contains either 1 or 2 keys and is hence either a 2-way or 3-way branch, respectively. Furthermore, all external nodes (i.e., leaves) are at the same level. Figure 1 shows some examples of 3-2 trees.

Insertion of a new key into a 3-2 tree is done as follows to preserve the 3-2 property: To add a new key into a node containing one key, simply insert it as the second key; if the node already contains two keys, split it into two one-key nodes and insert (recursively) the middle key into the parent node. This may cause the parent node to be split in a similar way, if it already contains two keys. For more details about 3-2 trees see [1] and [3].

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