$$(T_{2u+1})^2 + (T_{2u})^2 = T_{(2u+1)^2}$$

holds, we have the following theorem.

Theorem 10: The equation

$$T_{(2u+1)^2} = [(2u + 1)v]^2$$

with $v^2 = u^2 + (u + 1)^2$ has only a finite number of solutions.

Proof: Use Theorem 9.

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EXTENSIONS OF THE W. MNICH PROBLEM

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ABSTRACT

W. Sierpiński publicized the following problem proposed by Werner Mnich in 1956: *Are there three rational numbers whose sum and product are both one*? In 1960, J. W. S. Cassels proved that there are no rationals that meet the Mnich condition. This paper extends the Mnich problem to k-tuples of rationals whose sum and product are one by providing infinite solutions for all $k > 3$. It also provides generating forms that yield infinite solutions to the original Mnich problem in real and complex numbers, as well as providing infinite solutions for rational sums and products other than one.

HISTORICAL OVERVIEW

Sierpiński [6] cited a question posed by Werner Mnich as a most interesting problem, and one that at that time was unsolved. The Mnich question concerned the existence of three rational numbers whose sum and product are both one:

$$x + y + z = xyz = 1 \quad (x,\ y,\ z \text{ rational}). \tag{1}$$

Cassels [1] proved that there are no rationals that satisfy the conditions of (1). Cassels also shows that this problem was expressed by Mordell [3], in equivalent, if not exact form. Additionally, Cassels has compiled an excellent bibliography that demonstrates that the "Mnich" problem has its roots in the work of Sylvester [13] who in turn obtained some results from the 1870 work of the Reverend Father Pépin. Sierpiński [9] provides a more elementary proof of the impossibility of a weaker version of (1), along with an excellent summary of some of the equivalent forms of the "Mnich" problem. Later, Sansone and Cassels [4] provided another proof of the impossibility of (1).

EXTENSIONS TO k-TUPLES

It is natural to consider the generalization of the "Mnich" problem. Do there exist k-tuples of rational numbers such that their sums and products are both one for a given natural number k:

$$x_1 + x_2 + x_3 + \cdots + x_k = x_1x_2x_3 \ldots x_k = 1, \quad \text{for } k > 3? \tag{2}$$

Sierpiński [6, p. 127] states that Andrew Schinzel has proven that there are an infinite number of solutions for every k-tuple in (2). However, in the source cited, Trost [14] only appears to credit Schinzel with the proof of infinite k-tuples in (2) when k is of the form $4n$ or $4n + 1$, where n is a natural number (i.e., $k = 4, 5, 8, 9, 12, 13$, etc.). Schinzel provided a general form for generating an infinite number of solutions to (2) when $k = 4$. He provided one case for $k = 5$ (viz., $1, 1, 1, -1, -1$), but failed to demonstrate any solutions at all for (2) when the values of k are of the form $4n + 2$ or $4n + 3$ (viz., $6, 7, 10, 11, 14, 15$, etc.). Explicit generating functions will now be given which prove that there are indeed an infinite number of rational k-tuples for all $k > 3$ that satisfy (2).

It is quite obvious that for $k = 2$ there are no real solutions, since $xy = x + y = 1$ yields the quadratic equation $x^2 - x + 1 = 0$ whose discriminant is -3. For $k = 4$, a general form was given by Schinzel [14]:

$$\{n^2/(n^2 - 1),\ 1/(1 - n^2),\ (n^2 - 1)/n,\ (1 - n^2)/n\},\ n \neq \pm 1, 0, \tag{3}$$

e.g., for $n = 2$,

$$4/3 - 1/3 + 3/2 - 3/2 = (4/3)(-1/3)(3/2)(-3/2) = 1.$$

I derived the following general generating functions for all k-tuples greater than 4. Beyond the restrictions cited, they yield an infinite set of solutions for (2) by using any rational value of n.

$$\text{For } k = 5,\ \{n,\ -1/n,\ -n,\ 1/n,\ 1\},\ n \neq 0, \tag{4}$$

e.g., for $n = 2$,

$$2 - 1/2 - 2 + 1/2 + 1 = (2)(-1/2)(-2)(1/2)(1) = 1.$$

$$\text{For } k = 6,\ \{1/n^2(n + 1),\ -1/n^2(n + 1),\ (n + 1)^2,\ -n^2,\ -n,\ -n\},\quad n \neq 0, -1, \tag{5}$$

e.g., for $n = 2$,

$$1/12 - 1/12 + 9 - 4 - 2 - 2$$
$$= (1/12)(-1/12)(9)(-4)(-2)(-2) = 1.$$

$$\text{For } k = 7,\ \{(n - 1)^2,\ (n - 1/2),\ (n - 1/2),\ 1,\ -n^2,\ 1/n(n - 1)(n - 1/2),\ -1/n(n - 1)(n - 1/2)\},\quad n \neq 0, 1, 1/2, \tag{6}$$

e.g., for $n = 2$,

$$1 + 3/2 + 3/2 + 1 - 4 + 1/3 - 1/3$$
$$= (1)(3/2)(3/2)(1)(-4)(1/3)(-1/3) = 1.$$

Since the elements of the set $U = (1, -1, 1, -1)$ have a sum of 0 and a produce of 1, U forms the basis for generating all remaining explicit expressions beyond $k = 7$ by adjoining the elements of U onto the k-tuple results for $k = 4, 5, 6$, and 7. The process is then repeated as often as is necessary as a 4-cycle. For example:

(7) For $k = 8$,

$$\{n^2/n^2 - 1,\ n^2 - 1/n,\ 1 - n^2/n,\ 1/1 - n^2,\ 1, -1, 1, -1\}$$
$$= \{k = 4,\ U\},\ n \neq 0, -1, +1, \text{ or } \{k = 4,\ k = 4\}.$$

(8) For $k = 9$,

$$\{n, -1/n, -n, 1/n, 1, 1, -1, 1, -1\} = \{k = 5,\ U\},\ n \neq 0.$$

(9) For $k = 10$, $\{k = 6,\ U\}$; for $k = 11$, $\{k = 7,\ U\}$;
for $k = 12$, $\{k = 4,\ k = 4,\ k = 4\}$ or $\{k = 6,\ k = 6\}$
or $\{k = 8,\ U\}$ or $\{k = 8,\ k = 4\}$ or $\{k = 7,\ k = 5\}$
or $\{k = 4,\ U,\ U\}$.
Etc.

No claim is made here that the k-tuple form of the generating functions in (4) through (9) are unique.

EXTENSIONS TO OTHER NUMBER SYSTEMS

Although the conditions for generating rational roots for equation (1) have been demonstrated to be impossible, it is clear that rational roots approximating the Mnich criterion can be generated with any degree of accuracy required. Consider the example:

(10) $$(7/3)(-5/9)(-27/35) = 1, \text{ but } 7/3 - 5/9 - 27/35 = 951/845$$

and

(11) $$-.726547 - .540786 + 2.333333 = 1, \text{ but}$$
$$(-.726547)(-.5406786)(2.333333) = 0.9999999.$$

If the solution domain for equation (1) is expanded from the rationals to the reals, then there are an infinite number of solutions of the form $(a \pm \sqrt{b},\ c)$ which can be derived from the Mnich conditions

$$2a + c = 1 \quad\text{and}\quad (a^2 - b)c = 1.$$

One form of the solution in reals yields the following infinite set in which a is real:

(12) $$(a + \sqrt{a^2 + 1/(2a - 1)},\ a - \sqrt{a^2 + 1/(2a - 1)},\ 1 - 2a),\ a \neq 1/2,$$
e.g., for $a = 2$,
$$2 + \sqrt{13/3} + 2 - \sqrt{13/3} - 3 = (2 + \sqrt{13/3})(2 - \sqrt{13/3})(-3) = 1.$$

The generating form in (12) remains real as long as $a + 1/(2a - 1) \geq 0$, which is the case provided that $a > 1/2$ or $a \leq -.6572981$.

This latter condition for a makes the discriminant in (12) zero and yields the only solution with two equal elements:

(13) $$A = \left\{\sqrt[3]{-53/216 + \sqrt{13/216}} + \sqrt[3]{-53/216 - \sqrt{13/216}} + 1/6\right\};$$

$$B = 2\left\{1/3 - \sqrt[3]{-53/216 + \sqrt{13/216}} - \sqrt[3]{-53/216 - \sqrt{13/216}}\right\};$$

$$A + A + B = (A)(A)(B) = 1.$$

For values of a in the interval $-.6572981 < a < 1/2$, the generating form in (12) yields complex conjugate results, e.g.,

(14) $$a = 0,\ \{\pm i,\ 1\},\ \text{and}\ a = -1/2,\ \{(-1 \pm i)/2,\ 2\}.$$

Solutions of the Mnich problem in reals have not appeared in the literature, although Sierpiński [7, p. 176] does cite the first example in (14).

Also absent from the literature is a discussion of the Mnich problem in the complex plane. Assuming that the solution for (1) is of the form

$$(a \pm i\sqrt{b},\ c)$$

yields the infinite generating form with n real as follows:

(15) $$\left\{\frac{1 \pm i\sqrt{(n^3 - n + 2)/(n - 2)}}{n},\ \frac{n - 2}{n}\right\},\quad n \neq 0,\ 2,$$

e.g., for $a = 4$ $\{(1 \pm i\sqrt{31})/4,\ 1/2\}$.

The generating form in (15) remains complex as long as

$$(n^3 - n + 2)/(n - 2) \geq 0,$$

which is the case provided that $n > 2$ or $n < -1.5213797$. Note that these limits are the reciprocals of those for (12). When n is in the interval $-1.5213797 \leq n < 2$, (15) generates real solutions. Clearly, the generating forms (12) and (15) presented here for yielding real and complex solutions to (1) are not unique.

EXTENSIONS TO OTHER CONSTANT SUMS AND PRODUCTS

If the restriction in (2) that the product and sum must be equal to one is replaced by some rational number c, then a more general Mnich problem develops for rational x_i:

(16) $$x_1 + x_2 + x_3 + \cdots + x_k = x_1x_2x_3 \ldots x_k = c,\ \text{for}\ k \geq 2.$$

When $k = 2$, the infinite generating set is of the form:

(17) $$\{x,\ x/(x - 1)\}\ \text{where the product and sum} = x^2/(x - 1),\ x \neq 1,$$

e.g., $2 + 2 = (2)(2) = 4$, and $3 + 3/2 = (3)(3/2) = 9/2$.

When $k = 3$, then $x + y + z = xyz = c$. If we assume that $y = x/(x - 1)$ as in (17), then solving for z yields $z = (x + y)/(xy - 1) = x^2/(x^2 - x + 1)$. The infinite set is:

(18) $\left\{x, \dfrac{x}{x-1}, \dfrac{x^2}{x^2-x+1}\right\}$, $x \neq 1$, and the product and sum equal $x^4/(x^3 - 2x^2 + 2x - 1)$,

e.g., $2 + 2 + 4/3 = (2)(2)(4/3) = 16/3$.

When $k = 4$, using the previous results yields the infinite set:

(19) $\left\{x, \dfrac{x}{x-1}, \dfrac{x^2}{x^2-x+1}, \dfrac{x^4}{x^4 - x^3 - 2x^2 - 2x + 1}\right\}$, $x \neq 1$,

e.g., $2 + 2 + 4/3 + 16/13 = (2)(2)(4/3)(16/13) = 256/39$.

It is obvious that this process can be generalized in a recursive way to generate infinite rational solutions for any k-tuple.

Sierpiński [6, p. 127] credits Schinzel with demonstrating that the elements in (16) can be restricted to integers by the following substitutions for all $k \geq 2$:

(20) $x_{k-1} = 2$ and $x_k = k$ (fulfilled first), and

$x_1, x_2, x_3, \ldots, x_{k-2} = 1$.

The following table presents the results of (20).

k-Tuple	Solution Set	Product = Sum = C
2	(2, 2)	4
3	(1, 2, 3)	6
4	(1, 1, 2, 4)	8
5	(1, 1, 1, 2, 5)	10
⋮	⋮	⋮
k	$(\underbrace{1, 1, \ldots, 1,}_{k-2} 2, k)$	$k + 2 + \sum_{k=1}^{k-2} 1 = 2k$

It is worth noting that the number of integer solutions for sufficiently large k in (16) is still an open question. Also worth noting is that the result for $k = 3$ in the above table can be derived from assuming that the integers are of the form $x - p$, x, $x + p$, from which it follows that

$$x = \pm\sqrt{p^2 + 3}.$$

The only rational results generated are for $p = 1$ (1, 2, 3), and for $p = -1$ (-1, -2, -3).

SUMMARY

This paper traced the "Mnich" problem back to the work of Father Pépin in the 1870s, and identified the proofs of Cassels, Sansone, and Sierpiński as having decided the question in (1) in the negative. This restatement is needed because their results are not widely known, and sources such as (10) and (12) continue to cite the "Mnich" problem as unsolved.

Infinite generating forms for the extension of the Mnich conditions to all k-tuples greater than three are provided in (3)-(9). Infinite generating forms for the "Mnich" problem in the real and complex plane are provided in (12) and (15), respectively; also, approximate rational solutions are given in (10) and (11). Finally, the "Mnich" problem is extended to rational sums and products other than 1, and the recursive generating forms are provided for an infinite number of rational solutions for $k \geq 2$, with $k = 2$, $k = 3$, and $k = 4$ given explicitly, in (17)-(19).

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