Note the relationship to Stern numbers when expanding $f(m,n)$:

$$\begin{aligned} f(m,n) &= f(m,m+n) + f(m+n,n) \\ &= f(m,2m+n) + f(2m+n,m+n) \\ &\qquad + f(m+n,m+2n) + f(m+2n,n) \end{aligned}$$

The arguments of the function are generalized Stern numbers. The following conclusion can now be drawn concerning Eisenstein's function.

1. For any given $f(km + ln, k'm + l'n)$, that $(k + k')m + (l + l')n = \lambda$.
2. If $m = 1$ and $n = 2$, then (16) implies that $f(1, 2)$ can be composed of elements of the form $f(\propto, \lambda - \propto)$ and that
$$f(1, 2) = \lambda - \propto + \lambda - \propto' + \lambda - \propto'' + \cdots$$
3. For whole numbers "r" such that $\frac{\lambda + 1}{2} \le r \le \lambda - 1$,
$$f(1, 2) \equiv \sum \frac{1}{r} \pmod{\lambda}.$$
4. For whole numbers "r" such that, as in (18),
$$\frac{n_0\lambda}{n} \le r \le \frac{m_0\lambda}{m}$$
then
$$f(m,n) \equiv \sum \frac{1}{r} \pmod{\lambda}.$$

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A MULTINOMIAL GENERALIZATION OF A BINOMIAL IDENTITY

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1. The binomial identity which we wish to generalize is the following:

$$(x + y)^n = \sum_{k=1}^{n} \binom{2n-k-1}{n-1}\left(x^k + y^k\right)\left(\frac{xy}{x+y}\right)^{n-k}. \tag{1}$$

It can be found and is proved in [2]. Let us begin by giving a demonstration suitable to a generalization to more than two variables. Symbolizing $C_{t^n}f(t)$ for the coefficient a_n of t^n in any power series $f(t) = \sum_{n \ge 0} a_n t^n$, it is easily shown that the second number of (1) is:

(2) $$C_{t^{n-1}}\left(\frac{x}{1-tx}+\frac{y}{1-ty}\right)\left(1-t\frac{xy}{x+y}\right)^{-n}.$$

Indeed, it is sufficient to carry out the Cauchy product of the two following power series (in t):

$$\frac{x}{1-tx}+\frac{y}{1-ty}=\sum_{k\geq 1}(x^k+y^k)t^{k-1}$$

$$\left(1-t\frac{xy}{x+y}\right)^{-n}=\sum_{l\geq 0}\binom{n+l-1}{n-1}t^l\left(\frac{xy}{x+y}\right)^l.$$

To calculate (2) otherwise, let us apply the Lagrange reversion formula under the following form [1, I, p. 160 (8c)]: let $f(t)=\sum_{n\geq 0}a_n t^n$ be a formal series $a_0=0$, $a_1\neq 0$, of which the reciprocal series is $f^{\langle -1\rangle}(t)$ $\left[\text{that is to say, } f(f^{\langle -1\rangle}(t))=f^{\langle -1\rangle}(f(t))=t\right]$, and let $\Phi(t)$ be any other formal series with derivative $\Phi'(t)$; then we have:

(3) $$nC_{t^n}\,\Phi(f^{\langle -1\rangle}(t))=C_{t^{n-1}}\,\Phi'(t)\left(\frac{f(t)}{t}\right)^{-n}.$$

In view of demonstrating (1), let us put in (3),

$$f(t)=t-t^2\frac{xy}{x+y},\quad \Phi'(t)=\frac{x}{1-tx}+\frac{y}{1-ty},$$

which guarantees that the second member of (3) is effectively (2) in this case. But then,

$$\begin{aligned}\Phi(t)&=-\log(1-tx)-\log(1-ty)\\&=-\log\{1-t(x+y)+t^2xy\}\\&=-\log\{1-(x+y)f(t)\},\end{aligned}$$

that is to say, thanks to the well-known expansion $-\log(1-\tau)=\sum_{n\geq 1}\tau^n/n$ for (*):

$$\begin{aligned}nC_{t^n}\,\Phi(f^{\langle -1\rangle}(t))&=nC_{t^n}\,-\log\left\{1-(x+y)f(f^{\langle -1\rangle}(t))\right\}\\&=nC_{t^n}\,-\log(1-(x+y)t)\overset{(*)}{=}(x+y)^n;\end{aligned}$$

Consequently, we have equality (1) as a result of (3).

2. To generalize formula (1), let us call $\sigma_1, \sigma_2, \sigma_3, \ldots$ the *elementary* symmetric of the variables $x_1, x_2, \ldots, x_m$, and $S_1, S_2, S_3, \ldots$ the symmetric functions which are *sums of the powers*; in other words,

(4) $$\sigma_1=\sum_{1\leq i\leq m}x_i,\ \sigma_2=\sum_{1\leq i_1<i_2\leq m}x_{i_1}x_{i_2},\ \sigma_3=\sum_{1\leq i_1<i_2<i_3\leq m}x_{i_1}x_{i_2}x_{i_3},\ \ldots$$

(5) $$S_1\ (=\sigma_1)=\sum_{1\leq i\leq m}x_i,\ S_2=\sum_{1\leq i\leq m}x_i^2,\ S_3=\sum_{1\leq i\leq m}x_i^3,\ \ldots\ .$$

Let us apply the Lagrange formula (3), this time with

$$f(t) = t - t^2\frac{\sigma_2}{\sigma_1} + t^3\frac{\sigma_3}{\sigma_1} - \cdots + (-1)^{m-1}\frac{\sigma_m}{\sigma_1}t^m$$

$$= \frac{1}{\sigma_1}\left\{1 - (1 - tx_1)(1 - tx_2)\cdots(1 - tx_m)\right\},$$

$$\Phi(t) = -\log\bigl(1 - \sigma_1 f(t)\bigr) = -\log(1 - tx_1)(1 - tx_2)\cdots(1 - tx_m)$$

$$= -\sum_{j=1}^{m}\log(1 - tx_j),$$

$$\Phi'(t) = \frac{x_1}{1 - tx_1} + \frac{x_2}{1 - tx_2} + \cdots + \frac{x_m}{1 - tx_m}.$$

Now, the first member of (3) equals:

$$nC_{t^n} - \log\left\{1 - \sigma_1 f(f^{\langle -1\rangle}(t))\right\} = nC_{t^n} - \log(1 - \sigma_1 t) = \sigma_1^n, \tag{6}$$

and the second member of (3) may be written

$$C_{t^{n-1}}\ \Phi'(t)\left(\frac{f(t)}{t}\right)^{-n} \tag{7}$$
$$= C_{t^{n-1}}\left(\frac{x_1}{1 - tx_1} + \cdots + \frac{x_m}{1 - tx_m}\right)\left(1 - t\frac{\sigma_2}{\sigma_1} + t^2\frac{\sigma_3}{\sigma_1} - \cdots\right)^{-n}.$$

Let us introduce the simplified writing for the multinomial coefficients

$$(n - 1,\ \nu_1,\ \nu_2,\ \ldots,\ \nu_{m-1}) = \frac{(n - 1 + \nu_1 + \nu_2 + \cdots + \nu_{m-1})!}{(n-1)!\,\nu_1!\,\nu_2!\ \ldots\ \nu_{m-1}!}$$

$\left[\text{in particular, } (a, b - a) = \binom{b}{a}\right]$, and in expanding (7) as a multiple series of order $(m - 1)$, [1, I, p. 53 $(12m')$], there comes:

$$C_{t^{n-1}}\left\{\sum_{k\geq 1} S_k t^{k-1}\right\}\left\{\sum_{\nu_1,\nu_2,\ldots,\nu_{m-1}\geq 0}(n - 1,\ \nu_1,\ \nu_2,\ \ldots,\ \nu_{m-1})\,t^{\nu_1+2\nu_2+3\nu_3+\cdots}\left(\frac{\sigma_2}{\sigma_1}\right)^{\nu_1}\left(-\frac{\sigma_3}{\sigma_1}\right)^{\nu_2}\cdots\right\}. \tag{8}$$

Finally, by comparing (3), (6), and (8), we find:

<u>*Theorem*:</u> With the notations (4) and (5), we have the multinomial identity:

$$\sigma_1^n = \sum_{k=1}^{n}\left\{S_k \sum_{\nu_1+2\nu_2+\cdots+(m-1)\nu_{m-1}=n-k}(-1)^{\nu_2+\nu_4+\nu_6\cdots}(n - 1,\ \nu_1,\ \nu_2,\ \ldots)\left(\frac{\sigma_2}{\sigma_1}\right)^{\nu_1}\left(\frac{\sigma_3}{\sigma_1}\right)^{\nu_2}\cdots\left(\frac{\sigma_m}{\sigma_1}\right)^{\nu_{m-1}}\right\}. \tag{9}$$

For example, by $m = 2$, we find again formula (1) under the term

$$(x_1 + x_2)^n = \sum_{k=1}^{n} S_k \binom{2n-k-1}{n-1}\left(\frac{\sigma_2}{\sigma_1}\right)^{n-k}.$$

For three variables, x_1, x_2, x_3, $m = 3$, we have ($\nu = \nu_2$):

$$(x_1 + x_2 + x_3)^n = \sum_{k=1}^{n}\left\{S_k \sum_{0 \le \nu \le \frac{n-k}{2}} (-1)^\nu \frac{(2n-k-1-\nu)!}{(n-1)!\,\nu!\,(n-k-2\nu)!}\left(\frac{\sigma_2}{\sigma_1}\right)^{n-k-2\nu}\left(\frac{\sigma_3}{\sigma_1}\right)^{\nu}\right\}.$$

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WHICH SECOND-ORDER LINEAR INTEGRAL RECURRENCES HAVE ALMOST ALL PRIMES AS DIVISORS?

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This paper will prove that essentially only the obvious recurrences have almost all primes as divisors. An integer n is a divisor of a recurrence if n divides some term of the recurrence. In this paper, "almost all primes" will be taken interchangeably to mean either all but finitely many primes or all but for a set of Dirichlet density zero in the set of primes. In the context of this paper, the two concepts become synonymous due to the Frobenius density theorem. Our paper relies on a result of A. Schinzel [2], whose paper uses "almost all" in the same sense.

Let $\{w_n\}$ be a recurrence defined by the recursion relation

(1) $$w_{n+2} = aw_{n+1} + bw_n$$

where a, b, and the initial terms w_0, w_1 are all integers. We will call a and b the parameters of the recurrence. Associated with the recurrence (1) is its characteristic polynomial

(2) $$x^2 - ax - b = 0,$$

with roots α and β, where $\alpha + \beta = a$ and $\alpha\beta = -b$.

Let

$$D = (\alpha - \beta)^2 = a^2 + 4b$$

be the discriminant of this polynomial.

In general, if $D \neq 0$,

(3) $$w_n = c_1\alpha^n + c_2\beta^n,$$

where