4. W. E. Greig, "Bode's Rule and Folded Sequences," *The Fibonacci Quarterly* 14, No. 2 (1976):129-134.
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REFLECTIONS ACROSS TWO AND THREE GLASS PLATES

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1. INTRODUCTION

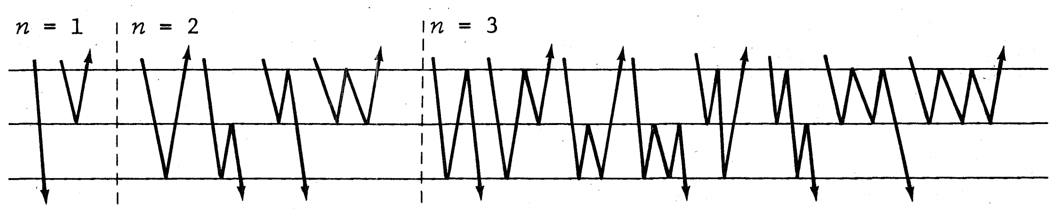
That reflections of light rays within two glass plates can be expressed in terms of the Fibonacci numbers is well known [Moser, 1]. In fact, if one starts with a single light ray and if the surfaces of the glass plates are half-mirrors such that they both transmit and reflect light, the number of possible paths through the glass plates with n reflections is F_{n+2}. Hoggatt and Junge [2] have increased the number of glass plates, deriving matrix equations to relate the number of distinct reflected paths to the number of reflections and examining sequences of polynomials arising from the characteristic equations of these matrices.

Here, we have arranged the counting of the reflections across the two glass plates in a fresh manner, fixing our attention upon the number of paths of a fixed length. One result is a physical interpretation of the compositions of an integer using 1's and 2's (see [3], [4], [5]). The problem is extended to three glass plates with geometric and matrix derivations for counting reflection paths of different types as well as analyses of the numerical arrays themselves which arise in the counting processes. We have counted reflections in paths of fixed length for regular and for bent reflections, finding powers of two, Fibonacci numbers and convolutions, and Pell numbers.

2. PROBLEM I

Consider the compositions of an even integer $2n$ into ones and twos as represented by the possible paths of length $2n$ taken in reflections of a light ray in two glass plates.

REFLECTIONS OF A LIGHT RAY IN PATHS OF LENGTH $2n$



For a path length of 2, there are 2 possible paths and one reflection; for a path length of 4, 4 possible paths and 8 reflections; for a path length of 6, 8 possible paths and 28 reflections. Notice that an odd path length would end at the middle surface rather than exiting.

First, the number of paths possible for a path length of $2n$ is easily derived if one notes that each path of length $2(n - 1)$ becomes a path of length $2n$ by adding a segment of length 2 which either passes through the center plate or reflects on the center plate, so that there are twice as many paths of length $2n$ as there were of length $2(n - 1)$.

Result 1: There are 2^n paths of length $2n$.

Continuing the same geometric approach yields the number of reflections for a path length $2n$. Each path of length $2(n - 1)$ gives one more reflection when a length 2 segment is added which passes through the center plate, and two more reflections when a length 2 segment is added which reflects on the center plate, or, the paths of length $2n$ have $3 \cdot 2^{n-1}$ new reflections coming from the 2^{n-1} paths of length $2(n - 1)$ as well as twice as many reflections as were in the paths of length $2(n - 1)$. Note that the number of reflections for path lengths $2n$ is $2^{n-1}(3n - 2)$ for $n = 1, 2, 3$. If there are

$$2^{n-2}(3(n - 1) - 2)$$

reflections in a path of length $2(n - 1)$, then there are

$$2 \cdot 2^{n-2}(3(n - 1) - 2) + 3 \cdot 2^{n-1} = 2^{n-1}(3n - 2)$$

reflections in a path of length $2n$, which proves the result following by mathematical induction.

Result 2: There are $2^{n-1}(3n - 2)$ reflections in each of the paths of length $2n$.

Proofs: Let A represent a reflection down or up, and B represent a straight path down or up, where both A and B have length two. Note that it is impossible for the two types of A to follow each other consecutively. Now, each path of length $2n$ is made up of A's and B's in some arrangement. Thus, the expansion of $(A + B)^n$ gives these arrangements counted properly, and $N = 2n$, so that the number of distinct paths is 2^n.

Now, in counting reflections, there is a built-in reflection for each A and a reflection between A and B, A and A, and B and B. Consider

$$f(x) = x^{n-1}(1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-1+j}.$$

Each term in $(A + B)^n$ has degree n and there are $(n - 1)$ spaces between factors. The x^{n-1} counts the $(n - 1)$ spaces between factors, since each A has a built-in reflection. The exponents of x count reflections from A; there are no reflections from B. Since we wish to count the reflections, we differentiate $f(x)$ and set $x = 1$.

$$\begin{aligned} f'(x) &= \left\{(n - 1)x^{n-2}(1 + x)^n + nx^{n-1}(1 + x)^{n-1}\right\}\Big|_{x=1} \\ &= (n - 1)2^n + n \cdot 2^{n-1} = 2^{n-1}(3n - 2). \end{aligned}$$

Interpretation as a composition using ones and twos: All the even integers have compositions in which, whenever strings of ones appear, there are an even number of them. Each A is a $1 + 1$ (taken as a pair) and each B is a 2, and each reflection is a plus sign. From $f(x)$, let $s = n - 1 + j$ so that $j = s - n + 1$, and we get $\binom{n}{s-n+1}$ compositions of $2n$, each with exactly s plus signs. Note that $s \geq n - 1$, with equality when all twos are used.

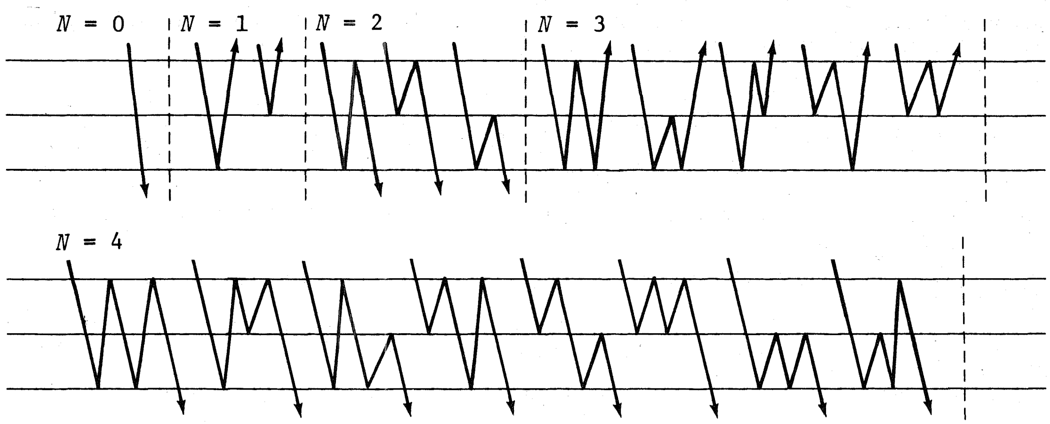
We note in passing that the number of possible paths through the two plates with n reflections is F_{n+2}, while the number of compositions of n using all ones and twos is F_{n+1} [3].

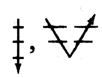
3. PROBLEM II

Given a particular configuration (path), how many times does it appear as a subconfiguration in all other paths with a larger but fixed number of reflections?

This leads to convolutions of the Fibonacci numbers.

PATHS WITH A FIXED NUMBER OF REFLECTIONS



Note that the subconfigurations  each occur 1, 2, 5, 10, 20, ... times in successive collections of all possible paths with a larger but fixed number of reflections. The same sequence occurs for any subconfiguration chosen.

Consider a subconfiguration that contains N reflections. It could be preceded by s reflections and followed by k reflections. Clearly, since each path starts at the upper left, the configurations in the front must start in the upper left and end up in the upper right, which demands an odd number of reflections. Thus, s is odd, but conceivably there are no configurations in

s	N reflections	k

the part on the front. Now, the part on the end could join up at the top or the bottom, depending on whether N is odd or even. In case N is even, then the regular configurations may be turned over to match. Thus, if the total number of reflections is specified, the allowable numbers will be determined.

4. RESULTS OF SEPARATING THE REFLECTION PATHS

In Sections 2 and 3, the reflection paths $\searrow$ and $\nearrow$, and $\wedge$ and $\vee$, were counted together. If one separates them, then, with the right side up, one obtains {1, 1, 4, 5, 14, 19, 46, 65, ...} which splits into two convolution sequences:

$$\{A_1, A_3, A_5, \ldots\} = \{1, 2, 5, 13, \ldots\} * \{1, 2, 5, 13, \ldots\};$$

$$\{A_2, A_4, A_6, \ldots\} = \{1, 2, 5, 13, \ldots\} * \{1, 3, 8, 21, \ldots\}.$$

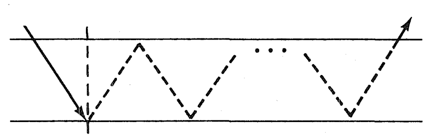
This second set agrees with the upside-down case {0, 1, 1, 5, 6, 19, 25, 65, ...} which splits into two convolution sequences:

$$\{B_1, B_3, B_5, \ldots\} = \{0, 1, 3, 8, \ldots\} * \{1, 3, 8, 21, \ldots\};$$

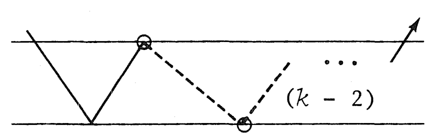
$$\{B_2, B_4, B_6, \ldots\} = \{A_2, A_4, A_6, \ldots\}.$$

Clearly, there are only two cases, $\searrow$, $\vee$, where we assume that the configurations in which these appear start at the left top and end at either right top or right bottom.

First we discuss the number of occurrences of $\searrow$. Here we consider only those patterns which start in the upper left. If there are no prepatterns, then we consider odd and even numbers of reflections separately. We get one free reflection by joining $\searrow$ to a pattern which begins on the bottom left.



Let us assume that the added-on piece has k (even) internal reflections. There are F_{k+2} such right-end pieces and $F_{0+2} = F_2 = 1$ left-end pieces. Next, let the piece on the right have $k - 2$ internal reflections and the one on the left have one internal reflection:



$$F_{1+2} \cdot F_{k-2+2}$$

Generally,

$$F_1F_{k+2} + F_3F_k + F_5F_{k-2} + \cdots .$$

Specifically,

$$k = 0: \quad F_1F_2 = 1$$

$$k = 2: \quad F_1F_4 + F_3F_2 = 1 \cdot 3 + 2 \cdot 1 = 5$$

$k = 4$: $F_1F_6 + F_3F_4 + F_5F_2 = 1 \cdot 8 + 2 \cdot 3 + 5 \cdot 1 = 19$

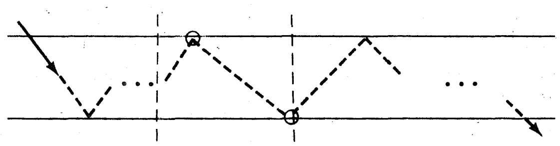
$k = 6$: $F_1F_8 + F_3F_6 + F_5F_4 + F_7F_2 = 1 \cdot 21 + 2 \cdot 8 + 5 \cdot 3 + 13 \cdot 1 = 65$

If k is odd, the same basic plan holds, so that for no pieces front or back, $F_2F_2 = 1$,

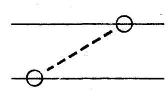
$k = 1$: $F_1F_3 + F_3F_1 = 1 \cdot 2 + 2 \cdot 1 = 4$

$k = 3$: $F_1F_5 + F_3F_3 + F_5F_1 = 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 = 14$

This is precisely the same as the other case except that it must start at the top left, have a free reflection where it joins a section at the top, a free reflection where it joins the right section at the bottom, and the right section must end at the bottom.



Any of our subconfigurations can appear complete by itself first. Our sample, of course, holds for any block with an even number of reflections. The foregoing depends on the final configuration starting on the upper left and the subconfiguration (the one we are watching) also starting on the upper left. However, if we "turn over" our subconfiguration then we get a different situation



which must fit into a standard configuration which starts in the upper left. Hence, this particular one cannot appear normally by itself, nor can any one with an even number of reflections. Here we must have a pre-configuration with an even number of reflections.

Let k be even again.

$F_2F_2 = 1 \cdot 1 = 1$

$F_2F_4 + F_4F_2 = 1 \cdot 3 + 3 \cdot 1 = 6$

$F_2F_6 + F_4F_4 + F_6F_2 = 1 \cdot 8 + 3 \cdot 3 + 8 \cdot 1 = 25$

...

Let k be odd.

$F_2F_1 = 1 \cdot 1 = 1$

$F_2F_3 + F_4F_1 = 1 \cdot 2 + 3 \cdot 1 = 5$

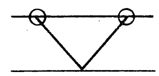
$F_2F_5 + F_4F_3 + F_6F_1 = 1 \cdot 5 + 3 \cdot 2 + 8 \cdot 1 = 19$

...

These sequences are $\{1, \underline{1}, 4, \underline{5}, 14, \underline{19}, \ldots\}$ (right side up) and $\{0, \underline{1}, 1, \underline{5}, 6, \underline{19}, \ldots\}$ (upside down), and added together, they produce the first Fibonacci convolution $\{1, 2, 5, 10, 20, 38, \ldots\}$.

Each subconfiguration which starts at the upper left and comes out at the lower right can be put in place of the configuration which makes a straight through crossing with the same results, of course.

For the results dealing with



the restrictions on the left are exactly the same as just described, and the endings on the right are merely those for the earlier case endings turned upside down to match the proper connection.

Reconsidering the four sequences of this section gives some interesting results. In the sequences $\{A_n\}$ (right side up) and $\{B_n\}$ (upside down), adding A_i and B_i gives successive terms of the first Fibonacci convolution sequence. Taking differences of odd terms gives $1-0=1$, $4-1=3$, $14-6=8$, ..., which is clearly $1, 3, 8, 21, \ldots, F_{2k}, \ldots$, the Fibonacci numbers with even subscripts.

Further, for $\{A_n\}$,

$$\begin{array}{ll} 1+1+2=4 & 1+4=5 \\ 4+5+5=14 & 5+14=19 \\ 14+19+13=46 & 19+46=65 \\ \cdots & \cdots \\ A_n+A_{n+1}+F_{n+2}=A_{n+2},\ n \text{ odd} & A_n+A_{n+1}=A_{n+2},\ n \text{ even} \end{array}$$

while for $\{B_n\}$,

$$\begin{array}{ll} 1+1+3=5 & 0+1=1 \\ 5+6+8=19 & 1+5=6 \\ 19+25+21=65 & 6+19=25 \\ \cdots & \cdots \\ B_n+B_{n+1}+F_{n+2}=B_{n+2},\ n \text{ even} & B_n+B_{n+1}=B_{n+2},\ n \text{ odd} \end{array}$$

The results of this section can be verified using generating functions as follows. (See, for example, [6].) The generating function for the first convolution of the Fibonacci sequence, which sequence we denote by $\{F_n^{(2)}\}$, is

$$\left(\frac{1}{1-x-x^2}\right)^2 = \sum_{n=0}^{\infty} F_{n+1}^{(2)}x$$

while the sequence of odd terms of $\{A_n\}$ is the first convolution of Fibonacci numbers with odd subscripts, or,

$$\left(\frac{1-x^2}{1-3x^2+x^4}\right)^2 = \sum_{n=0}^{\infty} A_{2n+1}x^n$$

and the sequence of odd terms of $\{B_n\}$ is the first convolution of Fibonacci numbers with even subscripts, or,

$$\left(\frac{x}{1-3x^2+x^4}\right)^2 = \sum_{n=0}^{\infty} B_{2n+1}x^n$$

and the even terms of $\{A_n\}$ as well as of $\{B_n\}$ are the convolution of the sequence of Fibonacci numbers with even subscripts with the sequence of Fibonacci numbers with odd subscripts, or,

$$\left(\frac{x}{1-3x^2+x^4}\right)\cdot\left(\frac{1-x^2}{1-3x^2+x^4}\right) = \sum_{n=0}^{\infty} A_{2n}x^n = \sum_{n=0}^{\infty} B_{2n}x^n.$$

That $\{F_n^{(2)}\}$ is given by the term-wise sum of $\{A_n\}$ and $\{B_n\}$ is then simply shown by adding the generating functions, since

$$\frac{(1-x^2)^2}{(1-3x^2+x^4)^2}+\frac{2x(1-x^2)}{(1-3x^2+x^4)^2}+\frac{x^2}{(1-3x^2+x^4)^2}$$

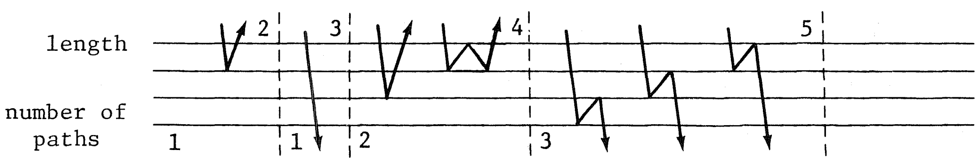
$$=\frac{(1+x-x^2)^2}{(1-3x^2+x^4)^2}=\frac{(1+x-x^2)^2}{(1-2x^2+x^4-x^2)^2}$$

$$=\frac{(1+x-x^2)^2}{(1-x^2+x)^2(1-x^2-x)^2}=\frac{1}{(1-x-x^2)^2}$$

Quite a few identities for the four sequences of this section could be derived by the same method.

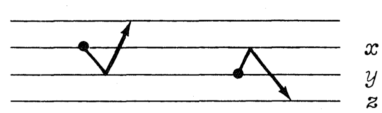
5. THREE STACKED PLATES

Theorem A: In reflective paths in three stacked glass plates, there are F_{n-1} paths of length n that enter at the top plate and exist at the top or bottom plate.



Discussion: Note that the paths end in lengths 3, 2 + 2, 1 + 1, or 1 + 2. We therefore assume of the paths of length n, that there are F_{n-4} which end in 3, F_{n-3} which end in 1 + 1, F_{n-5} which end in 2 + 2, F_{n-4} which end in 1 + 2, where $n \geq 5$. This is the same as saying that there are F_{n-4} paths of length $n - 3$ reflecting inwardly at an inside surface.

Proof: We proceed by induction. Thus the paths of length $k + 1$ are made up of paths which end in 3, 1 + 1, 2 + 2, or 1 + 2. We assume that there are F_{k-3} paths which end in 3, F_{k-2} paths which end in 1 + 1, F_{k-4} which end in 2 + 2, and F_{k-3} which end in 1 + 2. Since $F_{k-3} + F_{k-2} + F_{k-4} + F_{k-3} = F_k$, we will have a proof by induction if we can establish the assumption about path lengths. The first three are straightforward, but that F_{k-3} paths end in 1 + 2 needs further elaboration. In order to be on an outside edge after 1 + 2, the ray must have been on plate x or y with a reflection at the beginning:



How can the paths get to the x-dot for n even or the y-dot for n odd? Assume that there are F_{k-6} paths of length $k - 5$ which come from the upper surface, go to plate y, and then to the x-dot (note that the total path would then have length $k + 1$, since a path of 2 + 1 would be needed to reach the x-dot and a path of 1 + 2 to leave the x-dot). There are F_{k-5} paths which reflect from plate x, go to plate y and return to the x-dot, and F_{k-5} paths which relect from the bottom surface upward to the x-dot. Thus, there are

$$F_{k-6} + F_{k-5} + F_{k-5} = F_{k-3}$$

paths of length $k - 2$ coming upward to a reflection x-dot if k is even and downward to a y-dot if k is odd.

By careful counting, one can establish several other results involving Fibonacci numbers.

Theorem B: There are F_n paths of length n in three stacked plates that enter at the top plate and terminate on one of the internal surfaces.

Theorem C: There are F_{n+1} paths of length n which enter at the top plate and terminate on one of the four surfaces, and F_{n-1} that terminate on outside surfaces.

Theorem D: Of paths of length n terminating on any one of the four surfaces, there are F_n paths that end in a unit jump. There are $2F_{n-3}$ paths that end in a two unit jump, and there are F_{n-4} paths that end in a three unit jump.

Theorem E: There are nF_{n-3} ones used in all paths of length n which terminate on outside plates.

Theorem F: For $n \geq 3$, the number of threes in paths of length n which terminate on outside plates is a convolution of 1, 0, 1, 1, 2, 3, ..., F_{n-2}, ..., with itself. The convolution sequence is given by $2F_{n-4} + C_{n-6}$, where $C_n = (nL_{n+1} + F_n)/5$.

Theorem G: Let T_n' be the number of threes in all paths of length n that end on an inside line. Then the number of twos used in all paths of length n which terminate on outside faces is $2T_{n+1}' = 2F_{n-3} + 2C_{n-5}$.

Theorem H: $T_n' = T_n - F_{n-4}$, where T_n is the number of threes used totally in all paths of length n which terminate on outside faces, and T_n' is the number of threes in all paths of length n which end on an inside plate.

Corollary: The number of twos used in all paths of length n which terminate on outside surfaces is

$$2(T_{n+1} - F_{n-3}) = 2(2F_{n-3} + C_{n-5} - F_{n-3}) = 2[5F_{n-3} + (n-5)L_{n-4} + 2F_{n-5}]/5.$$

From this, of course, we can now discuss the numbers of ones, twos, and threes used in the reflections. We will let U_n be the number of ones used, D_n the number of twos, T_n the number of threes used in all paths of length n terminating on outside faces, while we will prime these to designate paths that only terminate on inside plates.

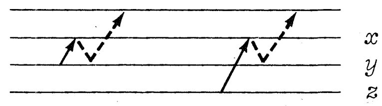
We return to the proof of Theorem A, that there are F_{n-1} paths of length n in three stacked glass plates, to glean more results. Recall that the plate paths end in 3, 1 + 1, 2 + 2, and 1 + 2.

Let P_n be the number of paths of length n. Then

$$P_n = P_{n-3} + P_{n-2} + P_{n-4} + \emptyset_{n-3},$$

where P_{n-3} paths end in 3, P_{n-2} in 1 + 1, P_{n-4} in 2 + 2, and $\emptyset_{n-3}$ is the number of paths terminating on an inside plate and of length n, but the last path segment was from the inside (i.e., from plate y to x). Suppose we approach x from below and the path is $n - 3$ units long; then we add the dotted portion. However, we can get to x from y or we can get to x from z. The number of paths from z is F_{n-6} by induction since there are F_{n-1} paths. The number of paths from y is $\emptyset_{n-4}$. Assume $\emptyset_n = F_{n-1}$ also so that

$$\emptyset_{n+1} = \emptyset_n + F_{n-2} = F_{n-1} + F_{n-2} = F_n.$$



Now

$$\begin{aligned} P_{n+1} &= P_{n-2} + P_{n-1} + P_{n-3} + \emptyset_{n-2} \\ &= (F_{n-3} + F_{n-2}) + (F_{n-4} + F_{n-3}) \\ &= F_{n-1} + F_{n-2} = F_n. \end{aligned}$$

If we display all F_{n-1} paths of length n, the number of ones used is nF_{n-3}.

We need some further results. Earlier we saw that there were F_{n-1} paths from the inside approaching one of the inside plates. We now need to know how many paths approach the inside lines from outside (a unit step from an outside line). Clearly, it is F_{n-2}; since the path length to the inside line is n, then the path length to the outside line is $n - 1$, making F_{n-2} paths. Let U_n be the number of ones used:

$$U_{n+1} = (U_{n-2} + 2U_{n-3}) + (U_{n-3}) + (U_{n-4}) + (U_{n-3} + U_{n-4}),$$

considering paths ending in 1 + 1, 3, 2 + 2, and 1 + 2.

Let us look at T_n, the number of threes used in paths of length n. By taking paths ending in 3, then 1 + 1, 2 + 2, and 1 + 2, we have

(A) $$T_n = (T_{n-3} + F_{n-4}) + T_{n-2} + T_{n-4} + T'_{n-3}$$

(B) $$T'_n = T'_{n-1} + T_{n-2}$$

Writing (A) for T_{n+1} and subtracting the expression above for T_n gives

$$\begin{aligned} T_{n+1} - T_n &= T_{n-1} - T_{n-4} + T'_{n-2} - T'_{n-3} + F_{n-3} - F_{n-4} \\ &= T_{n-1} + F_{n-5} + (T'_{n-2} - T'_{n-3} - T_{n-4}) \\ &= T_{n-1} + F_{n-5} + 0. \end{aligned}$$

Therefore,

$$T_{n+1} = T_n + T_{n-1} + F_{n-5},$$

which shows that $\{T_n\}$ is a Fibonacci convolution (first) sequence. It is easy to verify that

$$T_n = 2F_{n-4} + C_{n-6},\ T_1 = 0,\ T_2 = 0,\ T_3 = 1,$$
$$T_4 = 0,\ T_5 = 2,\ T_6 = 2,$$

where $\{C_n\}$ is *the* first Fibonacci convolution sequence.

Also,

$$T'_n = T_n - F_{n-4}$$

Next, consider D_n, the number of twos used in paths of length n. Again taking paths ending in 3, then in 1 + 1, 2 + 2, and 1 + 2, we have

(C) $$D_n = (D_{n-4} + 2F_{n-5}) + D_{n-3} + D_{n-2} + (D'_{n-3} + F_{n-4})$$

(D) $$D'_n = D'_{n-1} + D_{n-2} + F_{n-3}$$

Proceeding exactly as before, writing (C) for D_{n+1} and subtracting the expression for D_n, and then using identity (D), one derives

$$D_{n+1} = D_n + D_{n-1} + 2F_{n-4}.$$

We now show that $D_n = 2T'_{n+1}$. From $T'_n = T_n - F_{n-4}$, then

$$\begin{aligned}2T'_{n+2} &= 2T_{n+2} - 2F_{n-2}\\ &= 2T_{n+1} - 2F_{n-3} + 2T_n - 2F_{n-4} + 2F_{n-4}\end{aligned}$$

by taking advantage of $T_n = T_{n-1} + T_{n-2} + F_{n-6}$. Therefore,

$$T_{n+2} = T_{n+1} + T_n + F_{n-2} - F_{n-3} = T_{n+1} + T_n + F_{n-4}.$$

From the total length of F_{n-1} paths of length n, we know that

$$U_n + 2D_n + 3T_n = nF_{n-1},$$

so that

$$U_n = nF_{n-1} - 3T_n - 2D_n.$$

On the right-hand side, each term will satisfy a recurrence of the form

$$H_n = H_{n-1} + H_{n-2} + K_n,$$

where K_n is a generalized Fibonacci sequence. In this case, by looking at

$$U_1 = 0,\ U_2 = 2,\ U_3 = 0,\ U_4 = 4,$$

$$U_n = U_{n-1} + U_{n-2} + L_{n-4}.$$

This is precisely satisfied by $U_n = nF_{n-3}$.

If U_n is the number of ones used, D_n the number of twos, and T_n the number of threes, then clearly every number is followed by a reflection except the last one. Thus, if there are F_n total paths, then the number of reflections in paths of length n which terminate on outside faces is

$$\begin{aligned}R_n &= U_n + D_n + T_n - F_{n-1}\\ &= (nF_{n-3}) + \left(\frac{2}{5}[5F_{n-3} + (n-5)L_{n-4} + 2F_{n-5}]\right)\\ &\qquad + (2F_{n-4} + [(n-6)L_{n-5} + 2F_{n-6}]/5) - F_{n-1}\\ &= [(5n-3)F_{n-3} + (n-3)L_{n-2}]/5,\ n \ge 1.\end{aligned}$$

In summary, we write

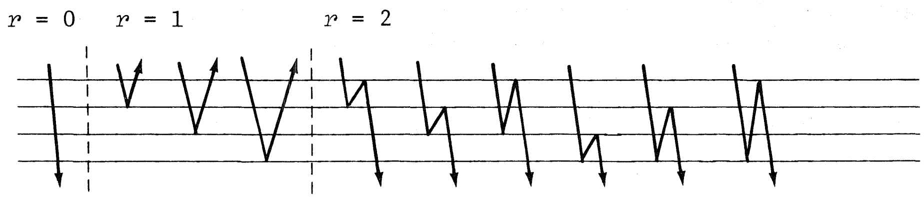
Theorem 1: In the total paths of length n which exit at outside plates, the number of paths is F_{n-1}, and the number of reflections R_n is

$$U_n + D_n + T_n - F_{n-1},$$

where

$$\begin{aligned}U_n &= nF_{n-3}\\ D_n &= \left(\frac{2}{5}[5F_{n-3} + (n-5)L_{n-4} + 2F_{n-5}]\right)\\ T_n &= 2F_{n-4} + [(n-6)L_{n-5} + 2F_{n-6}]/5\\ R_n &= [(5n-3)F_{n-3} + (n-3)L_{n-2}]/5.\end{aligned}$$

To conclude our discussion of paths and reflections in three glass plates, we consider a fixed number of reflections for paths which exit through either outside surface.



When there are $r=0$ reflections, there is 1 path possible; for $r=1$, 3 paths, and for $r=2$, 6 paths. The number of paths P_r for r reflections yields the sequence 1, 3, 6, 14, 31, 70, 157, ...

Theorem J: Let P_r be the number of paths which exit through either outside face in three glass plates and contain r reflections. Then

$$P_{r+1} = 2P_r + P_{r-1} - P_{r-2}$$

where

$$P_0 = 1,\ P_1 = 3,\ P_2 = 6,\ P_3 = 14.$$

It is easy to derive the sequence $\{P_r\}$. P_{r+1} is formed by adding a reflection at the outside face for each P_r path, and by adding a reflection at surface 1 or 2, which is the number of paths in P_r that end in a two unit jump plus twice the number ending in a three unit jump, which is P_{r-1}. The number ending in a unit jump in P_r paths is P_{r-2}. The number ending in a two unit jump in P_r paths is $P_r - P_{r-2} - P_{r-1}$. Thus,

$$\begin{aligned} P_{r+1} &= P \ + (P_r - P_{r-2} - P_{r-1}) + 2P_{r-1} \\ &= 2P_r + P_{r-1} - P_{r-2}. \end{aligned}$$

Fults [7] has given an explicit expression for P_r as well as its generating function.

6. A MATRIX APPROACH TO REFLECTIONS IN TWO AND THREE STACKED PLATES

Besides counting paths of constant length or paths of a constant number of reflections, there are many other problems one could consider. Here, matrices give a nice method for solving such counting problems.

We return to two glass plates and the paths of length n, where we consider paths that go from line zero to lines one and two, one step at a time. Let u_n, v_n, and w_n be the paths of length n to lines 0, 1, and 2, respectively, and consider the matrix Q defined in the matrix equation below, where we note that $QV_n = V_{n+1}$ and $Q^nV_1 = V_{n+1}$, as below:

$$QV_n = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = V_{n+1}$$

It is easy to see that $u_{n+1} = v_n$, since a path to line zero could have come only from line 1; therefore, each path to line zero was first a path of length n to line 1, then one more step to line zero. Paths to line 1 could have come from line zero or line two, so that $v_{n+1} = u_n + w_n$. Paths to line 2 came from line 1, or, $w_{n+1} = v_n$. This sets up the matrix Q whose characteristic polynomial is $x^2 - 2x = 0$ with solutions $x = 0$ or $x^2 = 2$, so that

$$u_{n+2} = 2u_n,\quad v_{n+2} = 2v_n,\quad \text{and}\quad w_{n+2} = 2w_n.$$

All paths of length zero start on line zero, and in one step of unit length one obtains only one path to line 1, or, using matrix Q,

$$QV_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = V_1$$

Sequentially, we see $Q^n V_0 = V_n$, or,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \to \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} \to \cdots \begin{pmatrix} 2^{n-1} \\ 0 \\ 2^{n-1} \end{pmatrix} \to \begin{pmatrix} 0 \\ 2^n \\ 0 \end{pmatrix} \cdots$$

Now, notice that there are 2^{n-1} paths coming out of the top line and 2^{n-1} paths coming out of the bottom line, each of length $2n$, so that there are 2^n such paths.

If one lets u_n^*, v_n^*, and w_n^* be the number of *regular reflections* on the paths of length n beginning on the top plate and terminating on the top, middle, or bottom plate, respectively, then it can be shown that, from the geometry of the paths,

$$\begin{aligned} u_{n+1}^* &= v_n^* + u_{n-1} \\ v_{n+1}^* &= u_n^* + w_n^* + 2v_{n-1} \\ w_{n+1}^* &= v_n^* + w_{n-1}. \end{aligned}$$

We can write both systems of equations in a 6×6 matrix

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \cdot \begin{pmatrix} u_n^* \\ v_n^* \\ w_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ u_n \\ v_n \\ w_n \end{pmatrix}.$$

The method of solution now can be through solving the system of equations directly and, once the recurrence relations are obtained, recognize them. Or one can work with the characteristic polynomial $[x(x^2-2)]^2$ via the Hamilton-Cayley theorem and go directly for the generating functions. The recurrence relations yield the general form of the generating function

$$\frac{r(x)}{P_n(x)} = A_0 + A_1 x + A_2 x^2 + \cdots,$$

whence one can get as many values as needed from the matrix application repeated to a starting column vector, as

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

to use the method of undetermined coefficients for $r(x)$.

The regular reflections are or , while the *bends* look like . These occur in paths which permit horizontal moves as well as jumps between surfaces. These are necessarily more complicated. The matrix Q^* yields paths of length n where "bend" reflections are allowed. That is,

$$Q^*V_n = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = V_{n+1}$$

allows paths to move along the lines themselves as well as between the lines. The same reasoning prevails. The characteristic polynomial $(1-x)(x^2-2x-1)$ yields Pell numbers for the paths of length n, sequentially, as

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \to \begin{pmatrix} 9 \\ 12 \\ 8 \end{pmatrix} \to \begin{pmatrix} 21 \\ 29 \\ 20 \end{pmatrix} \to \begin{pmatrix} 50 \\ 70 \\ 49 \end{pmatrix} \to \cdots$$

The formation of the number sequences themselves is easy, since

$$u_{n+1} = v_n + u_n, \quad w_{n+1} = u_{n+1} - 1, \quad \text{and} \quad v_{n+1} = 2v_n + v_{n-1}.$$

We see that paths of length n to line 1 are the Pell numbers P_n,

$$P_{n+1} = 2P_n + P_{n-1}, \; P_0 = 0, \; P_1 = 1,$$

while the paths to lines 0 and 2 have sums 1, 3, 7, 17, ..., the sum of two consecutive Pell numbers. In terms of Pell numbers P_n, we can write

$$u_n + w_n = P_n + P_{n-1} \quad \text{and} \quad u_n - w_n = 1,$$

so that

$$u_n = (P_n + P_{n-1} + 1)/2$$
$$v_n = P_n$$
$$w_n = (P_n + P_{n-1} - 1)/2.$$

This means that u_n and w_n separately obey the recurrence

$$U_{n+3} = 3U_{n+2} - U_{n+1} - U_n,$$

whose characteristic polynomial is

$$x^3 - 3x^2 + x + 1 = (x - 1)(x^2 - 2x - 1).$$

The corresponding matrix for the system with bend reflections is

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \cdot \begin{pmatrix} u_n^* \\ v_n^* \\ w_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ u_n \\ v_n \\ w_n \end{pmatrix}$$

Now, there are, of course, *regular reflections* along these paths, too, as well as bends, and the corresponding matrix for these is

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \cdot \begin{pmatrix} u_n^* \\ v_n^* \\ w_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ u_n \\ v_n \\ w_n \end{pmatrix}$$

with starting vector $u_1^* = v_1^* = w_1^* = 0$, $u_0 = 1$, $v_0 = w_0 = 0$.

One can verify that the generating functions for u_n^*, v_n^*, and w_n^* are

$$u_n^*: \quad \frac{(1-x)^4 + 2x^2}{(1-x)^2(1-2x-x^2)^2}$$

$$v_n^*: \quad \frac{3(1-x)^3x}{(1-x)^2(1-2x-x^2)^2}$$

$$w_n^*: \quad \frac{4(1-x)^2 - 2x^2}{(1-x)^2(1-2x-x^2)^2}$$

while their sum, $u_n^* + v_n^* + w_n^*$, yields the generating function

$$\frac{1 + x + 2x^2}{(1-2x-x^2)^2}$$

all clearly related to the Pell sequence, Pell first convolution, and partial sum of the Pell first convolution sequence.

In three stacked plates, these three systems of matrices generalize nicely. For *regular reflections* in paths of equal length n without horizontal moves,

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u_n^* \\ v_n^* \\ w_n^* \\ y_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ y_{n+1}^* \\ u_n \\ v_n \\ w_n \\ y_n \end{bmatrix},$$

while the *bend reflections* have the system

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{bmatrix} u_n^* \\ v_n^* \\ w_n^* \\ y_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ y_{n+1}^* \\ u_n \\ v_n \\ w_n \\ y_n \end{bmatrix},$$

and the *regular reflections in bent paths* are given by

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{bmatrix} u_n^* \\ v_n^* \\ w_n^* \\ y_n^* \\ u_{n-1} \\ v_{n-1} \\ w_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1}^* \\ v_{n+1}^* \\ w_{n+1}^* \\ y_{n+1}^* \\ u_n \\ v_n \\ w_n \\ y_n \end{bmatrix}.$$

7. REFLECTIONS ALONG BEND PATHS IN THREE STACKED PLATES

Here we count bend reflections and regular reflections in paths where bends are allowed. We begin with bend reflections in bend paths. Let U_n, V_n, W_n, and Y_n be the number of paths of length n terminating on lines 0, 1, 2, and 3, respectively. Let U_n^*, V_n^*, W_n^*, and Y_n^* be the number of bend reflections for those paths, and let a *bend* be a horizontal segment in a path. We shall show the following:

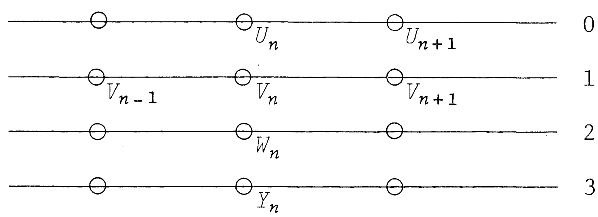
(A) $$U_{n+1}^* = V_n^* + U_n^* + 2V_{n-1}$$

(B) $V_{n+1}^* = V_n^* + U_n^* + W_n^* + 2(U_{n-1} + W_{n-1})$

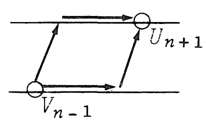
(C) $W_{n+1}^* = W_n^* + Y_n^* + V_n^* + 2(Y_{n-1} + V_{n-1})$

(D) $Y_{n+1}^* = Y_n^* + W_n^* + 2W_{n-1}$

We need a geometric derivation for the bends.



The paths to the point marked U_n contain U_n^* bends, and there are U_n such paths. We can go to U_{n+1} from V_{n-1} by either the upper or lower path, but we have added a bend at the upper path and a bend at the lower path;



thus, $2V_{n-1}$ merely counts the extra bends by these end moves. We can reach U_{n+1} from U_n and from V_n and each of these path bundles contains by declaration U_n^* and V_n^* bends, respectively. Thus,

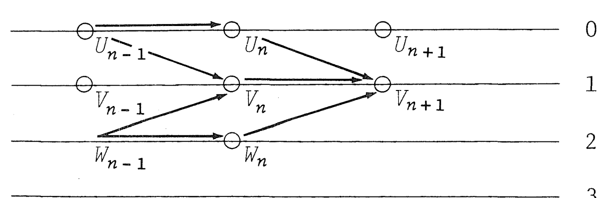
$$U_{n+1}^* = U_n^* + V_n^* + 2V_{n-1},$$

establishing (A). The derivation for (D) is similar.

We now tackle (B). Notice that we can reach V_{n+1} in a unit step from U_n, V_n, or W_n, so that we must count all bends in each of those previously counted paths, with no new bends added. We cannot use V_{n-1}, but paths routed through W_{n-1} and U_n or W_{n-1} and W_n as well as those through U_{n-1} and U_n or through U_{n-1} and V_n each collect one new bend, so that the number of added bends is $2(U_{n-1} + W_{n-1})$, making

$$V_{n+1}^* = U_n^* + V_n^* + W_n^* + 2(U_{n-1} + W_{n-1}),$$

which is identity (B). Similarly, we could establish (C).



To solve the system of equations (A), (B), (C), (D), let

$$A_n^* = U_n^* + Y_n^* \qquad A_n = U_n + Y_n$$

and

$$B_n^* = V_n^* + W_n^* \qquad\qquad B_n = V_n + W_n$$

Then (A) added to (D) yields

(E*) $$A_{n+1}^* = A_n^* + B_n^* + 2B_{n-1}$$

while (B) plus (C) yields

(G*) $$B_{n+1}^* = A_n^* + 2B_n^* + 2(A_{n-1} + B_{n-1}).$$

Let

$$C_n^* = U_n^* - Y_n^* \qquad\qquad C_n = U_n - Y_n$$

and

$$D_n^* = V_n^* - W_n^* \qquad\qquad D_n = V_n - W_n$$

Then subtracting (D) from (A) and (C) from (B) yields, respectively,

(F*) $$C_{n+1}^* = C_n^* + D_n^* + 2D_{n-1}$$

and

(H*) $$D_{n+1}^* = C_n^* + 2(C_{n-1} + D_{n-1}).$$

Now, A_n, B_n, C_n, and D_n are easily found. Returning to the first diagram of this section, from $U_{n+1} = U_n + V_n$ and $Y_{n+1} = Y_n + W_n$, we have

(E) $$A_{n+1} = A_n + B_n$$

(F) $$C_{n+1} = C_n + D_n$$

while $V_{n+1} = U_n + V_n + W_n$ and $W_{n+1} = W_n + V_n + Y_n$ yield

(G) $$B_{n+1} = 2B_n + A_n$$

(H) $$D_{n+1} = C_n.$$

From (E), we get $B_n = A_{n+1} - A_n$, which we use in (G) to obtain

$$(A_{n+2} - A_{n+1}) = 2(A_{n+1} - A_n) + A_n,$$

so that

$$A_{n+2} - 3A_{n+1} + A_n = 0.$$

From the starting data, $A_1 = 1$, $A_2 = 2$, so that A_n is a Fibonacci number with odd subscript, and

$$A_n = F_{2n-1}$$

$$B_n = A_{n+1} - A_n = F_{2n+1} - F_{2n-1} = F_{2n}.$$

From (F) and (H), in a similar manner, one finds that

$$C_n = F_{n+1} \quad \text{and} \quad D_n = F_n.$$

From these, we can find U_n, V_n, W_n, and Y_n by simultaneous linear equations, using

$$\begin{cases} U_n + Y_n = F_{2n-1} \\ U_n - Y_n = F_{n+1} \end{cases} \qquad \begin{cases} V_n + W_n = F_{2n} \\ V_n - W_n = F_n \end{cases}$$

The solutions are

$$\begin{cases} U_n = (F_{2n-1} + F_{n+1})/2 \\ Y_n = (F_{2n-1} - F_{n+1})/2 \end{cases} \qquad \begin{cases} V_n = (F_{2n} + F_n)/2 \\ W_n = (F_{2n} - F_n)/2 \end{cases}$$

Notice that

$$U_n + V_n + W_n + Y_n = F_{2n+1}.$$

Next, we can solve the full system for A_n^*, B_n^*, C_n^*, and D_n^*, since we now know A_n, B_n, C_n, and D_n. From (E*),

$$B_n^* = A_{n+1}^* - A_n^* - 2B_{n-1},$$

which substituted into (G*) gives us

$$(A_{n+2}^* - A_{n+1}^* - 2B_n) = A_n^* + 2(A_{n+1}^* - A_n^* - 2B_{n-1}) + 2(A_{n-1} + B_{n-1})$$

which simplifies to

$$A_{n+2}^* - 3A_{n+1}^* + A_n^* = 2B_n + 2A_{n-1} - 2B_{n-1} = 2L_{2n-2}$$

where we recognize the recursion relation for alternate Fibonacci numbers on the left while, as seen above, B_n and A_{n-1} are alternate Fibonacci numbers. It can be verified directly that if

$$A_n^* = 2(n-1)F_{2n-4},$$

then $A_{n+2}^* - 3A_{n+1}^* + A_n^* = 2L_{2n-2}$. From $B_n^* = A_{n+1}^* - A_n^* - 2B_{n-1}$ and $B_n = F_{2n}$ we get

$$B_n^* = 2nF_{2n-3} - 2F_{2n-3} = 2(n-1)F_{2n-3}.$$

In a similar fashion, we can verify that

$$C_{n+2}^* - C_{n+1}^* - C_n^* = 2L_n$$

is satisfied by

$$C_n^* = 2(n-1)F_{n-2}$$

and from

$$D_n^* = C_{n-1}^* + 2(C_{n-2} + D_{n-2})$$

where $C_n = F_{n+1}$ and $D_n = F_n$, we obtain

$$D_n^* = 2(n-1)F_{n-3}.$$

From these, we get

$$U_n^* = (n-1)(F_{2n-4} + F_{n-2})$$

$$V_n^* = (n-1)(F_{2n-3} + F_{n-3})$$

$$W_n^* = (n-1)(F_{2n-3} - F_{n-3})$$

$$Y_n^* = (n-1)(F_{2n-4} - F_{n-2})$$

This completes our solution for bend reflections in bend paths in three glass plates.

It is instructive, however, to consider a matrix approach to counting bend reflections in bend paths. A matrix which corresponds to the system of equations just given, counting the number of paths of length n and the number of bend reflections for those paths, is

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \cdot \begin{bmatrix} U_n^* \\ V_n^* \\ W_n^* \\ Y_n^* \\ U_{n-1} \\ V_{n-1} \\ W_{n-1} \\ Y_{n-1} \end{bmatrix} = \begin{bmatrix} U_{n+1}^* \\ V_{n+1}^* \\ W_{n+1}^* \\ Y_{n+1}^* \\ U_n \\ V_n \\ W_n \\ Y_n \end{bmatrix}$$

Expanding the characteristic polynomial,

$$\begin{aligned} &[(x-1)^4 - 3(x-1)^2 + 1]^2 \\ &= \left[((x-1)^2 - 1)^2 - (x-1)^2\right]^2 \\ &= [x^2 - 2x + 1 - 1 - (x-1)]^2[x^2 - 2x + 1 - 1 + (x-1)]^2 \\ &= (x^2 - 3x + 1)^2(x^2 - x - 1)^2 = 0 \end{aligned}$$

Notice that $(x^2 - 3x + 1) = 0$ yields the recurrence relation for the alternate Fibonacci numbers, while $(x^2 - x - 1) = 0$ gives the regular Fibonacci recurrence. A generating function derivation could be made for all formulas given in this section.

Values of the vector elements generated by the matrix equation for

$$n = 1, \ldots, 7$$

are given in the table below.

BEND REFLECTIONS

n	1	2	3	4	5	6	7
U_n^*	0	0	4	12	40	120	360
V_n^*	0	2	4	18	56	180	552
W_n^*	0	0	4	12	48	160	516
Y_n^*	0	0	0	6	24	90	300
U_{n-1}	1	1	2	4	9	21	51
V_{n-1}	0	1	2	5	12	30	76
W_{n-1}	0	0	1	3	9	25	68
Y_{n-1}	0	0	0	1	4	13	38

Finally, we list values for A_n^*, B_n^*, C_n^*, and D_n^*:

n	1	2	3	4	5	6	7	n
$A_n^* = U_n^* + Y_n^*$	0	0	4	18	64	210	660	$2(n-1)F_{2n-4}$
$B_n^* = V_n^* + W_n^*$	0	2	8	30	104	340	1068	$2(n-1)F_{2n-3}$
$C_n^* = U_n^* - Y_n^*$	0	0	4	6	16	30	60	$2(n-1)F_{n-2}$
$D_n^* = V_n^* - W_n^*$	0	2	0	6	8	20	36	$2(n-1)F_{n-3}$

We now shift our attention to the problem of counting regular reflections which occur in paths of length n in which bends are allowed. The matrix which solves the system of equations in that case follows, where starred entries denote regular reflections; otherwise, the definitions are as before. Notice that the characteristic polynomial is the same as that of the preceding matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \cdot \begin{bmatrix} U_n^* \\ V_n^* \\ W_n^* \\ Y_n^* \\ U_{n-1} \\ V_{n-1} \\ W_{n-1} \\ Y_{n-1} \end{bmatrix} = \begin{bmatrix} U_{n+1}^* \\ V_{n+1}^* \\ W_{n+1}^* \\ Y_{n+1}^* \\ U_n \\ V_n \\ W_n \\ Y_n \end{bmatrix}$$

Values of successive vector elements for $n = 1, \ldots, 8$ are given in the table following:

REGULAR REFLECTIONS IN BEND PATHS

n	1	2	3	4	5	6	7	8
U_n^*	0	1	2	7	20	60	176	517
V_n^*	0	0	3	9	31	95	290	868
W_n^*	0	0	0	5	20	75	250	794
Y_n^*	0	0	0	0	6	30	118	406
U_{n-1}	1	1	2	4	9	21	51	127
V_{n-1}	0	1	2	5	12	30	76	195
W_{n-1}	0	0	1	3	9	25	68	182
Y_{n-1}	0	0	0	1	4	13	38	106

The system of regular reflections in bend paths is not solved explicitly here, but generating functions for successive values are not difficult to obtain by using the characteristic polynomial of the matrix just given. Generating functions for A_n^*, B_n^*, C_n^*, and D_n^* are:

$$A_n^* = U_n^* + Y_n^*: \quad \frac{x^2(1 - 4x + 6x^2)}{(1 - 3x + x^2)^2}$$

$$B_n^* = V_n^* + W_n^*: \quad \frac{x^3(3 - 4x)}{(1 - 3x + x^2)^2}$$

$$C_n^* = U_n^* - Y_n^*: \quad \frac{x^2(1 + 2x^2)}{(1 - x - x^2)^2}$$

$$D_n^* = V_n^* - W_n^*: \quad \frac{x^3(3 - 2x)}{(1 - x - x^2)^2}$$

Since $A_n^* + B_n^* = U_n^* + V_n^* + W_n^* + Y_n^*$, the generating function for regular reflections in bend paths terminating on all four surfaces is

$$\frac{(x^2 - x^3 + 2x^4)}{(1 - 3x + x^2)^2}.$$

8. REGULAR REFLECTIONS IN THREE STACKED PLATES

If one wishes equations for the number of paths ending upon certain lines and the number of regular reflections, the procedure is the same as when "bends" are allowed, as in the last section. Let U_n, V_n, W_n, and Y_n be the number of paths of length n from line 0 to lines 0, 1, 2, and 3. Let U_n^*, V_n^*, W_n^*, and Y_n^* be the number of regular reflections counted for those paths.

The system of equations to solve is

$$\begin{aligned} U_{n+1}^* &= V_n^* + U_{n-1} & \qquad U_{n+1} &= V_n \\ V_{n+1}^* &= U_n^* + W_n^* + 2V_{n-1} & V_{n+1} &= U_n + W_n \\ W_{n+1}^* &= Y_n^* + V_n^* + 2W_{n-1} & W_{n+1} &= V_n + Y_n \\ Y_{n+1}^* &= W_n^* + Y_{n-1} & Y_{n+1} &= W_n \end{aligned}$$

These differ from the equations used in Section 7 only in that no horizontal moves along the lines are allowed, so that one represses terms that correspond to that same line. The method of solution is exactly the same.

One finds that

$$\begin{aligned} U_{2k} &= F_{2k-1} & \qquad U_{2k+1} &= 0 \\ Y_{2k+1} &= F_{2k} & Y_{2k} &= 0 \\ V_{2k+1} &= F_{2k+1} & V_{2k} &= 0 \\ W_{2k} &= F_{2k} & W_{2k+1} &= 0 \end{aligned}$$

which agrees with Theorems A and B of Section 5, since $U_n + Y_n = F_{n-1}$ is the number of paths ending at outside lines, while $V_n + W_n = F_n$ is the number of paths ending on inside surfaces. Notice that $U_n + V_n + W_n + Y_n = F_{n+1}$, which agrees with Theorem C.

As for the number of reflections to paths ending on outside surfaces,

$$U_n^* = C_{n-1} - 2C_{n-2} + 3C_{n-3},\ n \text{ even};\ U_n^* = 0,\ n \text{ odd};$$

$$Y_n^* = C_{n-1} - 2C_{n-2} + 3C_{n-3},\ n \text{ odd};\ \ Y_n^* = 0,\ n \text{ even};$$

where $\{C_n\}$ is the first Fibonacci convolution, $C_n = (nL_{n+1} + F_n)/5$. One can verify that the total number of reflections for paths of length n which exit at either outside surface is $U_n^* + Y_n^* = C_{n-1} - 2C_{n-2} + 3C_{n-3}$, which is equivalent to the formula given for R_n in Theorem I of Section 5.

Finally, we write, again for the first Fibonacci convolution $\{C_n\}$,

$$V_n^* = 3C_{n-2} - C_{n-3},\ n \text{ odd};\ \ V_n^* = 0,\ n \text{ even};$$

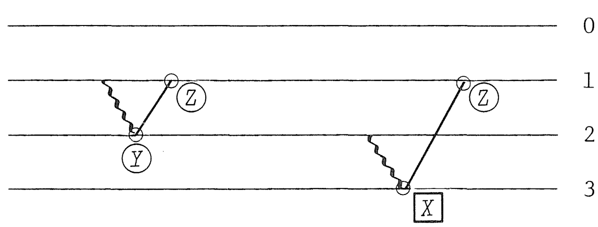
$$W_n^* = 3C_{n-2} - C_{n-3},\ n \text{ even};\ W_n^* = 0,\ n \text{ odd}.$$

Here, the matrix solution for the number of regular reflections in paths without bends follows from

$$\left[\begin{array}{cccc:cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \cdot \begin{bmatrix} U_n^* \\ V_n^* \\ W_n^* \\ Y_n^* \\ U_{n-1} \\ V_{n-1} \\ W_{n-1} \\ Y_{n-1} \end{bmatrix} = \begin{bmatrix} U_{n+1}^* \\ V_{n+1}^* \\ W_{n+1}^* \\ Y_{n+1}^* \\ U_n \\ V_n \\ W_n \\ Y_n \end{bmatrix}.$$

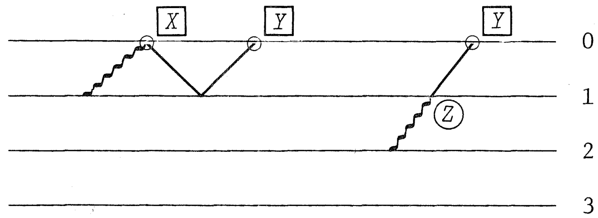
9. NUMERICAL ARRAYS ARISING FROM REGULAR REFLECTIONS IN THREE STACKED PLATES

Let circled numbers denote reflections on paths coming to the inside lines from the inside. Let boxed numbers denote reflections in paths to the outside lines.



$$Z = Y + \boxed{X}$$

Note that Z is one longer and one reflection more than Y, while it is two longer and one reflection more than X. Since the paths under discussion are to the inside lines from the inside, paths going from 2 to 1 imply a reflection as indicated. Since the paths from 3 must have come from 2, this also implies a reflection as shown. Thus, $Z = Y + \boxed{X}$. Secondly, the two types of reflections are related by $\boxed{Y^*} = Z^* + \boxed{X^*}$ from considering the following:



$\boxed{Y} = Z + \boxed{X}$

Paths indicated which come through from the inside are extended to Y by one but do not add a reflection. The paths coming through which have one added reflection at the inside line imply a reflection at X since paths to the top line can come only from the middle line.

The geometric considerations just made give the recursive patterns in the following array. The circled numbers are the number of reflections for paths of length n which enter from the top and terminate on inside lines by segments crossing the center space only (*not* immediately reflected from either outside face), while the boxed numbers are regular paths from the top line to either outside line.

Reflections

Path Length	0	1	2	3	4	5	6	7	...
1									
2	1	$\boxed{1}$							
3	$\boxed{1}$	1							
4		$\boxed{1}$	2	$\boxed{1}$					
5		1	$\boxed{3}$	2					
6		$\boxed{1}$	2	$\boxed{3}$	3	$\boxed{1}$			
7			$\boxed{2}$	5	$\boxed{6}$	3			
8			1	$\boxed{6}$	8	$\boxed{6}$	4	$\boxed{1}$	
...									
	$\boxed{A}$			$\boxed{A^*}$					
	B	$\boxed{C}$		B^*					
	$\boxed{D}$				C^*				
			$\boxed{E}$						

$\boxed{E} = \boxed{A} + B + \boxed{C} + \boxed{D}$ $C^* = B^* + \boxed{A^*}$

Please note that each row sum is $2F_{n-1}$, where the sum of the circled numbers as well as the sum of the boxed numbers is in each case F_{n-1}. Also note that the row sum is not the total number of paths of length n, since, for example, when $n = 5$, there is one path with two reflections which terminates inside, and one path with four reflections which terminates inside. Also note that the circled diagonal numbers in the table are partial sums of the boxed diagonal numbers in the diagonal above.

Let $D_n(x)$ be the generating function for the nth diagonal sequence going downward to the right in the table. That is, $D_0(x)$ generates the boxed sequence 1, 0, 1, 0, 1, 0, 1, ... and $D_1(x)$ generates the circled sequence 1, 1, 2, 2, 3, 3, 4, 4, ..., while $D_2(x)$ generates the boxed sequence 1, 1, 3, 3, 6, 6, ... From the table recurrence, $C^* = B^* + A^*$, since C^* and B^* are on the same falling diagonal,

$$D_1(x) = x^2 D_1(x) + D_0(x),$$

so that

$$D_1(x) = [D_0(x)]/(1 - x^2).$$

We write

$$D_0(x) = \frac{x}{1 - x^2}$$

$$D_1(x) = \frac{1 + x}{(1 - x^2)^2}$$

$$D_2(x) = \frac{1 + x}{(1 - x^2)^3}$$

$$D_3(x) = \frac{(1 + x)^2}{(1 - x^2)^4}$$

$$D_4(x) = \frac{(1 + x)^2}{(1 - x^2)^5}$$

Notice that $D_n(x)$ generates boxed numbers for n even and circled numbers for n odd. Summing $D_n(x)$ for n even gives the row sum for the boxed numbers by producing the generating function for the Fibonacci sequence and, similarly, for taking n odd and circled numbers. The column sums of circled or boxed numbers each obey the recurrence $u_n = 2u_{n-1} + u_{n-2} - u_{n-3}$.

Notice that

$$D_{2n+1}(x) = [D_1(x)]^{n+1}$$

$$D_{2n}(x) = (1 - x)D_{2n+1}(x) = (1 - x)[D_1(x)]^{n+1},$$

so we see once again the pleasantry of a convolution array intimately related to Pascal's triangle.

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ON PSEUDO-FIBONACCI NUMBERS OF THE FORM $2S^2$, WHERE S IS AN INTEGER

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If the pseudo-Fibonacci numbers are defined by

$$u_1 = 1,\ u_2 = 4,\ u_{n+2} = u_{n+1} + u_n,\ n > 0, \tag{1}$$

then we can show that $u_1 = 1$, $u_2 = 4$, and $u_4 = 9$ are the only square pseudo-Fibonacci numbers.

In this paper we will describe a method to show that none of the pseudo-Fibonacci numbers are of the form $2S^2$, where S is an integer.

Even if we remove the restriction $n > 0$, we do not obtain any number of the form $2S^2$, where S is an integer.

It can be easily shown that the general solution of the difference equation (1) is given by

$$u_n = \frac{7}{5.2^n}(\alpha^n + \beta^n) - \frac{1}{5.2^{n-1}}(\alpha^{n-1} + \beta^{n-1}), \tag{2}$$