MAXIMUM CARDINALITIES FOR TOPOLOGIES ON FINITE SETS

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If $[n]$ represents the first n natural numbers, D. Stephen showed in [3] that no topology on $[n]$ with the exception of the discrete topology has more than $3(2^{n-2})$ elements and that this number is a maximum. In this article we show that, if k is a nonnegative integer and $k \leq n$, then no topology on $[n]$ with precisely $n - k$ open singletons has more than $(1 + 2^k)2^{n-k-1}$ elements and that this number is attainable over such topologies for $k < n$. We also show that the topology on $[n]$ with no open singletons and the maximum number of elements has cardinality $1 + 2_{n-2}$.

Recently, A. R. Mitchell and R. W. Mitchell have given a much simpler proof of Stephen's result [2]. Their proof consists of showing (1) If $n \geq 2$ and $x,y \in [n]$ with $x \neq y$, then

$$\Gamma(x,y) = \{A \subset [n]: x \in A \text{ or } y \notin A\}$$

is a topology on $[n]$ with precisely $3(2^{n-2})$ elements, and (2) If Γ is a nondiscrete topology on $[n]$, there exist $x,y \in [n]$ with $\Gamma \subset \Gamma(x,y)$. In Section 1, we give proofs of two theorems which in conjunction produce Stephen's result and which dictate what form the nondiscrete topology of maximum cardinality must have.

1. STEPHEN'S RESULT

We let $|A|$ denote the cardinality of a set A. If Γ is a topology on $[n]$ and $x \in [n]$, we let $M(\Gamma,x)$ be the open set about x with minimum cardinality. Evidently, $\Gamma = \{A \subset [n]: M(\Gamma,x) \subset A \text{ whenever } x \in A\}$.

Theorem 1.1: If k is a positive integer and Γ is a topology on $[n]$ with precisely $n - k$ open singletons, there is a topology Δ on $[n]$ with precisely $n - k + 1$ open singletons and $|\Gamma| < |\Delta|$.

Proof: Choose $x \in [n]$ such that $\{x\}$ is not open. Let

$$\Delta = \Big\{A \cup (B \cap \{x\}): A,B \in \Gamma\Big\}.$$

Then Δ is a topology on $[n]$ with precisely $n - k + 1$ open singletons, which satisfies $\Gamma \subset \Delta$ and $\Gamma \neq \Delta$. The proof is complete.

Theorem 1.2: If k is a positive integer and Γ is a topology on $[n]$ with precisely $n - k$ open singletons and for some $x \in [n]$, $\{y\}$ is open for each

$$y \in M(\Gamma,x) - \{x\} \quad\text{and}\quad |M(\Gamma,x)| > 2,$$

there is a topology Γ on $[n]$ with precisely $n - k$ open singletons satisfying $|\Gamma| < |\Delta|$.

Proof: Choose $y \in M(\Gamma,x) - \{x\}$ and let

$$\Delta = \Big\{A \cup (B \cap (M(\Gamma,x) - \{y\})): A,B \in \Gamma\Big\}.$$

Then Δ is a topology on $[n]$ with precisely $n - k$ open singletons, which satisfies $\Gamma \subset \Delta$ and $\Gamma \neq \Delta$. The proof is complete.

Corollary 1.3: Each nondiscrete topology on $[n]$ has at most $3(2^{n-2})$ elements and this number is a maximum.

Proof: If Γ is a nondiscrete topology on $[n]$, then $n \geq 2$. From Theorem 1.1, if Γ has the maximum cardinality over all nondiscrete topologies on $[n]$, then Γ has precisely $n-1$ open singletons; and by Theorem 1.2, if $\{n\}$ is the non-open singleton, we must have $|M(\Gamma,n)| = 2$. So there is an $x \in [n-1]$ with $M(\Gamma,n) = \{n, x\}$. Thus,

$$\Gamma = \left\{A \subset [n]: n \notin A\right\} \cup \left\{A \subset [n]: \{n, x\} \subset A\right\}.$$

Consequently, $|\Gamma| = 2^{n-1} + 2^{n-2} = 3(2^{n-2})$ and the proof is complete.

Remark 1.4: The topology Δ in the proof of Theorem 1.1 (1.2) is known as the simple extension of Γ through the subset $\{x\}$ $(M(\Gamma,x) - \{y\})$ [1].

2. SOME PRELIMINARIES

In this section we present some notation and prove a theorem which will be useful in reaching our main results. If $k \in [n]$, let $\lambda(k)$ be the collection of topologies on $[n]$ which have $\{1\}$, $\{2\}$, ..., $\{k\}$ as the nonopen singletons. If $1 \leq m \leq k$, let $C(m)$ be the set of increasing functions from $[m]$ to $[k]$; for each $g \in C(m)$, let

$$U(\Gamma,m,g) = \bigcup_{i \in [m]} M(\Gamma,g(i))$$

and

$$\Omega(\Gamma,m,g) = \{A \subset [n]: U(\Gamma,m,g) \subset A \text{ and } |A \cap [k]| = m\}.$$

Lemma 2.1: The following statements hold for each topology $\Gamma \in \lambda(k)$.

(a) $\Gamma = \{A \subset [n]: A \cap [k] = \emptyset\} \cup \bigcup_{m=1}^{k} \bigcup_{g \in C(m)} \Omega(\Gamma,m,g)$.

(b) For each $m \in [k]$ and $g \in C(m)$, we have

$$|\Omega(\Gamma,m,g)| = 0 \quad \text{or} \quad |\Omega(\Gamma,m,g)| = 2^{n-k+m-|U(\Gamma,m,g)|}.$$

(c) $(\Gamma,m,g) \cap \Omega(\Gamma,j,h) = \emptyset$ unless $(m,g) = (j,h)$.

Proof of (a): Let Δ represent the set on the right-hand side of the equality sign in (a), and let $W \in \Gamma$. If $W \cap [k] = \emptyset$, then $W \in \Delta$. If $W \cap [k] \neq \emptyset$, then $|W \cap [k]| = m$ for some $m \in [k]$. Let g be the strictly increasing function from $[m]$ to $W \cap [k]$. For each $g(i)$ we have $W \supset M(\Gamma,g(i))$, so

$$W \supset U(\Gamma,m,g), \quad W \in \Omega(\Gamma,m,g), \text{ and } \Gamma \subset \Delta.$$

If $W \in \Delta$ and $W \cap [k] = \emptyset$, then $W \in \Gamma$. Otherwise, $W \in \Omega(\Gamma,m,g)$ for some $m \in [k]$ and $g \in C(m)$. For this (m,g) we have

$$g([m]) \subset U(\Gamma,m,g) \subset W;$$

thus, $W \in \Gamma$, since

$$W = U(\Gamma,m,g) \cup (W - U(\Gamma,m,g)), \quad U(\Gamma,m,g) \in \Gamma,$$

and

$$(W - U(\Gamma,m,g)) \cap [k] = \emptyset;$$

so $\Delta \subset \Gamma$ and (a) is verified.

Proof of (b): It is easy to verify that $\Omega(\Gamma,m,g)$ is the set of all subsets of $[n] - ([k] - g([m]))$ which contain $U(\Gamma,m,g)$ for each pair (m,g). Consequently (b) holds.

Proof of (c): If $A \in \Omega(\Gamma,m,g) \cap \Omega(\Gamma,j,h)$, then $m = |A \cap [k]| = j$. Also,

$$g([m]) \cup h([m]) \subset A \cap [k],$$

which gives

$$|g([m]) \cup h([m])| = m.$$

Since g and h are strictly increasing, we must have $g = h$, and the proof is complete.

We are now in a position to establish the following useful theorem.

Theorem 2.2: If Γ is an element of $\lambda(k)$, then

$$|\Gamma| \leq 2^{n-k} + \sum_{m=1}^{k} \sum_{g \in C(m)} 2^{n-k+m-|U(\Gamma,m,g)|}$$

with equality if and only if $\Omega(\Gamma,m,g) \neq \emptyset$ for any pair (m,g).

Proof: From Lemma 2.1(a) and (c), we have

$$|\Gamma| = |\{A \subset [n]: A \cap [k] = \emptyset\}| + \sum_{m=1}^{k} \sum_{g \in C(m)} |\Omega(\Gamma,m,g)|.$$

So from Lemma 2.1(b) we get

$$|\Gamma| \leq 2^{n-k} + \sum_{m=1}^{k} \sum_{g \in C(m)} 2^{n-k+m-|U(\Gamma,m,g)|}$$

with equality if and only if $\Omega(\Gamma,m,g) \neq \emptyset$ for any pair (m,g). The proof is complete.

3. THE FIRST TWO OF OUR MAIN RESULTS

The Case $0 \leq k \leq n$: The results are clear for $k = 0$. In the following, we assume that $k \in [n]$.

Theorem 3.1: If n is a positive integer and $\Gamma \in \lambda(k)$, then

$$|\Gamma| \leq (1 + 2^k)2^{n-k-1}.$$

Proof: We proceed by induction on n. The case $n = 1$ is true vacuously. Suppose $n > 1$ and the result holds for all integers $j \in [n - 1]$.

Case 1: $|U(\Gamma,m,g)| = m$ for some pair (m,g). Then we have

$$U(\Gamma,m,g) \subset [k].$$

Let $W \in \Gamma$ with $W \subset [k]$ and $|W|$ a minimum. Then $|W| \geq 2$ and $M(\Gamma,x) = W$ for each $x \in W$. Without loss, assume that $1 \in W$ and if $[n] - W \neq \emptyset$, assume that $[n] - W = \{2, 3, \ldots, n - |W| + 1\}$. Define a topology Δ on $[n - |W| + 1]$ by the following family of minimum-cardinality open sets:

$$M(\Delta,1) = \{1\},\ M(\Delta,x) = \big(M(\Gamma,x) - W\big) \cup \{1\} \text{ if } M(\Gamma,x) \cap W \neq \emptyset$$

and

$$M(\Delta,x) = M(\Gamma,x) \text{ otherwise.}$$

It is not difficult to show that $|\Delta| = |\Gamma|$ and that Δ has $n - k + 1$ open singletons. So by the induction hypothesis, we have

$$|\Gamma| \leq \big(1 + 2^{k-|W|}\big)2^{n-k} \leq (1 + 2^k)2^{n-k-1}.$$

Case 2: $|U(\Gamma,m,g)| > m$ for each pair (m,g). Here we have

$$|U(\Gamma,m,g)| \geq m+1$$

for each pair (m,g) and, from Theorem 2.2, we get

$$|\Gamma| \leq 2^{n-k} + \sum_{m=1}^{k} \sum_{g \in C(m)} 2^{n-k+m-|U(\Gamma,m,g)|} \leq 2^{n-k} + \left(\sum_{m=1}^{k} \binom{k}{m}\right) 2^{n-k-1};$$

we see easily that

$$2^{n-k} + \left(\sum_{m=1}^{k} \binom{k}{m}\right) 2^{n-k-1} = (1+2^k)2^{n-k-1}.$$

The proof is complete.

It is obvious that if $\Gamma \in \lambda(k)$ with $|U(\Gamma,m,g)| = m+1$ for each pair (m,g) then $|\Gamma|$ will be a maximum over $\lambda(k)$ and we will have

$$|\Gamma| = (1+2^k)^{n-k-1}.$$

If such a Γ has $|\Gamma|$ a maximum over $\lambda(k)$, we must have

$$|M(\Gamma,x)| = 2 \quad \text{and} \quad |M(\Gamma,x) \cap [k]| = 1$$

for each $x \in [k]$, since $g \in C(1)$ defined by $g(1) = x$ must satisfy

$$|U(\Gamma,1,g)| = 2 \quad \text{and} \quad \Omega(\Gamma,1,g) \neq \emptyset$$

from Lemma 2.1(b). Moreover, if $x < y$ and $x,y \in [k]$, then

$$|M(\Gamma,x) \cup M(\Gamma,y)| = 3$$

since $g \in C(2)$ defined by $g(1) = x$ and $g(2) = y$ must satisfy

$$|U(\Gamma,2,g)| = 3.$$

Thus,

$$M(\Gamma,x) \cap M(\Gamma,y) \neq \emptyset.$$

This implies that there must be a $j \in [n] - [k]$ with $M(\Gamma,x) = \{x, j\}$ for each $x \in [k]$ and that

$$\Gamma = \{A \subset [n]: A \cap [k] = \emptyset\} \cup \left\{A \subset [n]: \{x, j\} \subset A \text{ for each } x \in A \cap [k]\right\}.$$

We have

$$|\Gamma| = (1+2^k)2^{n-k-1}$$

from the arguments above and the second of our main results is realized.

Theorem 3.2: For $0 \leq k < n$, there is a topology on $[n]$ with precisely $n-k$ open singletons and $(1+2^k)2^{n-k-1}$ elements.

As a by-product of these main results, we obtain Stephen's result.

Corollary 3.3: The only topology on $[n]$ having more than $3(2^{n-2})$ open sets is the discrete topology. Moreover, this upper bound cannot be improved.

Proof: If the topology Γ on $[n]$ is not discrete, then $n > 1$ and there is at least one nonopen singleton. If k is the number of nonopen singletons, we have, from Theorem 3.1, that

$$|\Gamma| \leq 2^{n-1} + 2^{n-k-1} \leq 2^{n-1} + 2^{n-2} = 3(2^{n-2}),$$

and since $n \neq 1$, there is a topology on $[n]$ with precisely $3(2^{n-2})$ elements, from Theorem 3.2. The proof is complete.

4. OUR FINAL TWO MAIN RESULTS

The Case $k = n$: It is obvious that for $k = n$, no topology on $[n]$ has

$$(1 + 2^k)2^{n-k-1}$$

elements. If $\Gamma \in \lambda(n)$, we let

$$\Phi(\Gamma) = \{A \subset [n]: A = M(\Gamma,x) \text{ for each } x \in A, \text{ and } \neq \emptyset\}.$$

It is clear from the argument in Case 1 of Theorem 3.1 that $\Phi(\Gamma) \neq \emptyset$.

Theorem 4.1: If Γ is an element of $\lambda(k)$ which has maximum cardinality over $\lambda(k)$, then $|A| = 2$ for each $A \in \Phi(\Gamma)$.

Proof: If $A \in \Phi(\Gamma)$ with $|A| > 2$, choose $x,y \in A$ with $x \neq y$ and let

$$\Delta = \left\{V \cup (B \cap \{x, y\}): V,B \in \Gamma\right\}.$$

Then $\Delta \in \lambda(k)$, $\Gamma \subset \Delta$, and $\Gamma \neq \Delta$. The proof is complete.

Theorem 4.2: If Γ is an element of $\lambda(n)$, then $|\Gamma| \leq 1 + 2^{n-2}$.

Proof: Let $\Gamma \in \lambda(n)$ with $|\Gamma|$ a maximum. Then $|A| = 2$ for each $A \in \Phi(\Gamma)$. For each $i \in [|\Phi(\Gamma)|]$, let

$$P(i) = \{n - 2|\Phi(\Gamma)| + i,\ n - i + 1\};$$

without loss, assume that

$$\Phi(\Gamma) = \{P(i): i \in [|\Phi(\Gamma)|]\}$$

and that

$$[n] - \bigcup_{\Phi(\Gamma)} A = [n - 2|\Phi(\Gamma)|] \quad \text{if} \quad n \neq 2|\Phi(\Gamma)|.$$

Define a topology Δ on $[n - |\Phi(\Gamma)|]$ by specifying its minimum-cardinality open sets for each $x \in [n - |\Phi(\Gamma)|]$ as

$$M(\Delta,x) = \left(M(\Gamma,x) - \bigcup_{\Phi(\Gamma)} A\right) \cup \{n - 2|\Phi(\Gamma)| + i: P(i) \cap M(\Gamma,x) \neq \emptyset\}.$$

Then Δ has precisely $|\Phi(\Gamma)|$ open singletons and $|\Gamma| = |\Delta|$. By Theorem 3.1,

$$|\Gamma| \leq \left(1 + 2^{n-2|\Phi(\Gamma)|}\right)2^{|\Phi(\Gamma)|-1}$$

where the expression on the right side of the inequality decreases as $|\Phi(\Gamma)|$ increases. Thus, $|\Gamma| \leq 1 + 2^{n-2}$ for all $\Gamma \in \lambda(n)$ and the proof is complete.

Theorem 4.3: For $n > 1$, there is a topology on $[n]$ with no open singletons and $1 + 2^{n-2}$ elements.

Proof: From Theorem 3.2, there is a topology Γ on $[n-1]$ with $1 + 2^{n-2}$ elements. For this topology, $M(\Gamma,x) = \{x, n - 1\}$ for $x \neq n - 1$ and $M(\Gamma, n - 1) = \{n - 1\}$ may be assumed to be the minimum-cardinality open sets. Let

$$\Delta = \left\{A \subset [n]: M(\Gamma,x) \cup \{n\} \subset A \text{ when } M(\Gamma,x) \subset A\right\}.$$

Then Δ is a topology on $[n]$ with no open singletons and $|\Delta| = |\Gamma|$. The proof is complete.

5. SOME FINAL REMARKS

The following observations may be made from the Theorems and constructions above.

Remark 5.1: It is easy to construct for each $1 \leq j \leq n - k$ a topology $\Gamma \varepsilon \lambda(k)$ with cardinality $(2^k + (-1 + 2^j))2^{n-k-j}$. Let $M(\Gamma, x) = \{x\}$ for each $x \varepsilon [n] - [k]$ and $m(\Gamma, x) = \{x, k+1, k+2, \ldots, k+j\}$ for each $x \varepsilon [k]$. We see from Theorem 2.1 that $|\Gamma|$ is the required number.

Remark 5.2: More generally, if $k \varepsilon [n]$ and for each $x \varepsilon [k]$, $W(x)$ is a nonempty subset of $[n] - [k]$, let Γ be the topology on $[n]$ having minimal cardinality open sets $M(\Gamma, x) = \{x\} \cup W(x)$ for $x \varepsilon [k]$ and $M(\Gamma, x) = \{x\}$ otherwise. Then from Theorem 2.1

$$|\Gamma| = 2^{n-k} + \sum_{m=1}^{k} \sum_{g \varepsilon C(m)} 2^{n-k+m-\left(m + \left|\bigcup_{[m]} W(g(i))\right|\right)}$$

since

$$|U(\Gamma, m, g)| = \left|\bigcup_{[m]} M(\Gamma, g(i))\right| = |g([m])| + \left|\bigcup_{[m]} W(g(i))\right| = m + \left|\bigcup_{[m]} W(g(i))\right|.$$

Remark 5.3: For each $k \varepsilon [n]$, let

$$\mu(k) = \{\Gamma \varepsilon \lambda(k) : \Omega(\Gamma, m, g) \neq \emptyset \text{ for any pair } (m, g)\}.$$

Then $\mu(k) = \{\Gamma \varepsilon \lambda(k) : \text{for each } x \varepsilon [k],\ M(\Gamma, x) = \{x\} \cup W(x) \text{ for some nonempty } W(x) \subset [n] - [k]\}$. Thus $|\mu(k)| = (-1 + 2^{n-k})^k$ for each subset of $[n]$ of cardinality k. Therefore,

$$\binom{n}{k}(-1 + 2^{n-k})^k$$

is the number of topologies, Γ, on $[n]$ such that

$$\Gamma \varepsilon \lambda(k) \quad \text{and} \quad \Omega(\Gamma, m, g) \neq \emptyset \quad \text{for any pair } (m, g).$$

The total number of such topologies is

$$\sum_{k \varepsilon [n]} \binom{n}{k}(-1 + 2^{n-k})^k.$$

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