# A MODIFICATION OF GOKA'S BINARY SEQUENCE

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ABSTRACT

Goka's binary sequence $G = \left(g_i\right)_{i=1}^{\infty}$, where $g_i$ is a binary has been modified by replacing the binaries $(g_i)$ by matrices of the same order over the binaries. We define, formulate, and discuss the properties of the $n$th integral from $j$ of $G$ by repeating in succession Melvyn B. Nathanson's formula for $I_j G$, the integral from $j$ of $G$. The integral equation $I_j G = G$ has been solved. We investigate the behavior of the decimated sequence, submatrix sequence, sequence of integrals from $j$, and complementary sequence of the binary matrix sequence (BMS) $G$ in relation to $G$. An application of the binary sequence has been described.

## 1. INTRODUCTION

Goka [1] has introduced the binary sequence $G = \left(g_i\right)_{i=1}^{\infty}$, where $g_i = 0$ or 1 and the addition is modulo 2. Nathanson [2] has discussed eventually periodic binary sequences. In his paper he has formulated the $n$th derivative of $G$, $D^n G$, and the integral from $j$ of $G$, $I_j G$.

In this paper we present a modification of the binary sequence $G$ to the binary matrix sequence by replacing the binary $g_i$ by a $m \times n$ matrix over the binaries. All arithmetic of the binaries is done modulo 2, and the addition of the binary matrices or binary matrix sequences is done componentwise. Not surprisingly, we will find that all the results established in [2] hold good for our BMS also. We generalize the integration formula for $I_j G$, where $G$ is a BMS, formulate $I_j^n G$, the $n$th integral from $j$ of $G$, and establish results illustrating its properties. We study certain interesting properties of the decimated sequence of $G$, the submatrix sequence of $G$, and the complement of $G$ in relation to their parent BMS $G$. In the final section, we show how to apply the novel method of binary sequences to represent any sequence of integers. Just indicating whether a member is odd or even and using this method we are able to determine whether

$$\binom{p+r+1}{r}, \binom{p+r}{r}, \quad r = 0, 1, \ldots, n-1$$

are odd or even when $p = 2^m q$, where $q$ is odd and $2^{m-1} < n \leq 2^m$.

## 2. NOTATIONS AND DEFINITIONS

*Definition 1*: A binary matrix sequence (BMS) is the infinite sequence

$$G = \left(g_i\right)_{i=1}^{\infty},$$

where $(g_i)$ are matrices of the same order over the binaries.

In what follows, we will use the following laws of addition modulo 2:

(i) $1 + 0 = 0 + 1 = 1$;
(ii) $1 + 1 = 0 + 0 = 0$.

It is evident that, if $k$ is a nonnegative integer,

$$k\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\0\end{pmatrix},$$

according as $k$ is odd or even. Addition of the binary matrices and, likewise, the BMSs is done componentwise, i.e., if

$$g_i = (a_{rs}),\ g_j = (b_{rs}),\ g_i + g_j = (a_{rs} + b_{rs}).$$

Similarly, if

$$G = \left(g_i\right)_{i=1}^{\infty} \quad \text{and} \quad H = \left(h_i\right)_{i=1}^{\infty}$$

are two BMSs of the same order, then $G + H = \left(g_i + h_i\right)_{i=1}^{\infty}$.

*Notations*: $\mathbf{0}$, $\underset{\sim}{\mathbf{0}}$, $\boldsymbol{g}$, $\underset{\sim}{\boldsymbol{g}}$

(i) $\mathbf{0}$ = a binary matrix in which every entry is 0 and is called a binary null matrix.
(ii) $\underset{\sim}{\mathbf{0}}$ = a binary matrix sequence in which every entry is $\mathbf{0}$ and is called a constant null sequence.
(iii) $\boldsymbol{g}$ = a binary matrix in which every entry is 1.
(iv) $\underset{\sim}{\boldsymbol{g}}$ = a binary matrix sequence in which every entry is $\boldsymbol{g}$.

The following results will be useful. If $g_i$ is a binary matrix and $k$ is a nonnegative integer, then

(i) $kg_i = g_i$ or $\mathbf{0}$ according as $k$ is odd or even.
(ii) $g_i + g_j = \mathbf{0}$ means $g_i = g_j$.

*Definition 2*:

(i) If $g_i$ and $h_i$ are two binary matrices of the same order with $g_i + h_i = \boldsymbol{g}$, then each is said to be the complement of the other.
(ii) If $G$ and $H$ are two BMSs of the same order with $G + H = \underset{\sim}{\boldsymbol{g}}$, each sequence is said to be the complement of the other.

We use the notation $\overline{g}_i$, $\overline{G}$ for the complements of $g_i$, $G$, respectively, and to write the complement, we simply change the binaries 0, 1 to 1, 0, respectively.

*Example*: If $g_i = \begin{pmatrix}1&1\\0&1\\0&0\end{pmatrix}$, $g_j = \begin{pmatrix}1&0\\0&0\\1&1\end{pmatrix}$, then

(i) $$g_i + g_j = \begin{pmatrix}0&1\\0&1\\1&1\end{pmatrix}$$

(ii) $$3g_i = \begin{pmatrix}1&1\\0&1\\0&0\end{pmatrix} = g_i$$

(iii) $$8g_i = \begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} = \mathbf{0}$$

(iv) $\bar{g}_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

*Definition 3:*

(i) If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS, its derivative $DG = \left(g_i^1\right)_{i=1}^{\infty}$, where $g_i^1 = g_i + g_{i+1}$.

(ii) If $G$ is a BMS, its $n$th derivative is defined recursively by $D^nG = D(D^{n-1}G)$.

*Definition 4:*

(i) If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS, the integral from $j$ of $G$ is the BMS $I_jG$ whose $i$th term is

$$g_{i,1} = \begin{cases} \sum_{s=i}^{j-1} g_s & \text{if } i < j \\ \mathbf{0} & \text{if } i = j \\ \sum_{s=j}^{i-1} g_s & \text{if } i > j \end{cases}$$

(ii) If $G$ is a BMS, the $n$th integral from $j$ of $G$ is the BMS $I_j^nG$ defined recursively by $I_j^nG = I_j(I_j^{n-1}G)$.

*Definition 5:*

(i) If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS, the sequence of integrals from $j$ of $G$ is defined as $J = \left(I_j^nG\right)_{n=0}^{\infty}$, where $I_j^0G = G$.

(ii) The truncated sequence of integrals from $j$ up to $p$ is the infinite sequence $J_T$ whose $n$th term is the truncated BMS

$$\left(I_j^n\left(g_i\right)\right)_{i=1}^{p}.$$

*Definition 6:* If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS and $d$ is a positive integer, the decimated BMS $G^d$ of $G$ is defined by

$$G^d = \left(g_{kd}\right)_{k=1}^{\infty}.$$

*Definition 7:* $G^*$ is called a sequence of submatrices of a BMS $G$, and is obtained by taking the submatrices of the same location from the binary matrices of $G$.

*Example:*

If $G = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots;$

$$\left(G^d\right)_{d=2} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots;$$

$$G^* = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots;$$

obtained by taking second and third rows of the binary matrices of $G$.

*Definition 8:* Eventual property of a BMS.

A BMS $G = \left(g_i\right)_{i=1}^{\infty}$ is said to have an eventual property from $j$ when it it true for the BMS $\left(g_i\right)_{i=j}^{\infty}$.

## SECTION 3

In this section, we establish certain theorems with regard to the integrals from $j$ of a BMS.

*Theorem 1:* If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS, its $n$th integral from $j$ is the BMS $I_j^nG$ whose $i$th term $g_{i,n}$ is

(1.1) $\displaystyle\sum_{s=i}^{j-1}\binom{n-1+s-i}{n-1}g_s$ if $i < j$

(1.2) $\mathbf{0}$ if $j \le i < j+n$

(1.3) $\displaystyle\sum_{s=j}^{i-n}\binom{i-s-1}{n-1}g_s$ if $i \ge j+n$

*Proof:*

Case (1.1): Let $i < j$. Then $g_{i,n}$, the $i$th term in $I_j^nG$ is

$$\sum_{s=i}^{j-1}\left(\sum_{s_{n-1}=1}^{s}\sum_{s_{n-2}=1}^{s_{n-1}}\cdots\sum_{s_2=1}^{s_3}\sum_{s_1=1}^{s_2}1\right)g_{i+s-1}.$$

Upon using

$$\sum_{i_n=1}^{p}\sum_{i_{n-1}=1}^{i_n}\cdots\sum_{i_2=1}^{i_3}\sum_{i_1=1}^{i_2}1 = \binom{n+p-1}{n},$$

we have the $i$th term as

$$\sum_{s=i}^{j-1}\binom{n-1+s-i}{n-1}g_s.$$

Case (1.2): Let $j \le i < j+n$ and $I_j^nG = \left(g_{i,n}\right)_{i=1}^{\infty}$. As

$$g_{j,1} = \mathbf{0},\ g_{j,2} = g_{j+1,2} = \mathbf{0},\ g_{j,3} = g_{j+1,3} = g_{j+2,3} = \mathbf{0},$$

and finally,

$$g_{j,n} = g_{j+1,n} = g_{j+2,n} = \cdots = g_{j+n-1,n} = \mathbf{0}$$

or, in brief,

$$g_{i,n} = \mathbf{0} \text{ if } j \le i < j+n.$$

Case (1.3): Let $i \ge j+n$. Here the formula can be established on par with case (1.1) and the proof is therefore left to the reader.

*Corollary 1*: If $G = \left(g\right)_{i=1}^{\infty}$ is a constant BMS,

(1.4) (i) $D^n G = \underset{\sim}{\mathbf{0}}$

(ii) $I_j^n G = \left(g_{i,n}\right)_{i=1}^{\infty}$, where $g_{i,n}$ is:

(1.5) $$\binom{n+j-i-1}{n}g \quad \text{if } i < j$$

(1.6) $$\mathbf{0} \quad \text{if } j \le i < j+n$$

(1.7) $$\binom{i-j}{n}g \quad \text{if } i \ge j+n.$$

*Proof*: (1.4) follows immediately from the definition of the operator $D$, and (1.6) is a particular case of (1.2).

To prove (1.5), consider

$$g_{i,n} = \sum_{s=i}^{j-1}\binom{s-i+n-1}{n-1}g = \left(\sum_{s=i}^{j-1}\binom{s-i+n-1}{n-1}\right)g.$$

Upon using

$$\sum_{s=0}^{p}\binom{n+s}{n} = \sum_{s=0}^{p}\binom{n+s}{s} = \binom{n+p+1}{p},$$

$$g_{i,n} = \binom{n+j-i-1}{j-i-1}g = \binom{n+j-i-1}{n}g.$$

Similarly, we prove (1.7).

*Corollary 2*:

(i) $\underset{n\to\infty}{\mathrm{Lt}}\ I_j^n G$ is eventually null from $j$. This follows from (1.6).

(ii) $\underset{n\to\infty}{\mathrm{Lt}}\ I_1^n G = \underset{\sim}{\mathbf{0}}$.

*Theorem 2*: The truncated sequence of integrals $J_T$ from $j$ of a BMS $G\ (\ne \underset{\sim}{\mathbf{0}})$ up to $j - 1$ has a period $2^m$, where $2^{m-1} + 1 < j \le 2^m + 1$.

*Proof*: If $I^p G = \left(g_{i,p}\right)_{i=1}^{\infty}$ and $g_{i,p} = g_i$ for $i < j$ with $p$ in its lowest form then

$$g_{i,n+p} = g_{i,n} \text{ for } i < j \text{ and } n = 1, 2, 3, \ldots,$$

and hence it is sufficient to investigate the feasibility of

$$g_{i,p} = g_i,\ i < j.$$

This will happen when

$$\sum_{s=i}^{j-1}\binom{p+s-i-1}{p-1}g_s = g_i,\ i < j,$$

which is true when $\binom{p}{p-1}, \binom{p+1}{p-1}, \ldots, \binom{p+j-3}{p-1}$ are all even. The suitable value of $p = 2^m q$, where $q$ is odd and $m$ is nonnegative with $j \le 2^m + 1$. In order that $p$ is lowest, we set $q = 1$, $j > 2^{m-1} + 1$, and conclude that $J_T$ up to $j - 1$ of $J$ is of period $2^m$, where $2^{m-1} + 1 < j \le 2^m + 1$.

*Corollary*: If a BMS $G$ is eventually null from $j$, the sequence of integrals from $j$ of $G$ is of period $2^m$, where $2^{m-1} + 1 < j \le 2^m + 1$.

*Example*: Choosing $j = 5$, consider

$$G = \left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \ldots\right)$$

$$I_5^4 G = \left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \ldots\right)$$

From our choice of $j$, $2^m = 4$ and we find that $J_T$ up to 4 of $J$ has a period 4.

*Theorem 3*: If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS and $I_j G = G$, then $G = \underset{\sim}{\mathbf{0}}$.

*Proof*: If $I_j G = G$,

$$\sum_{s=i}^{j-1} g_s = g_i \quad \text{if} \quad i < j, \tag{3.1}$$

$$g_j = \mathbf{0} \quad \text{and} \quad \sum_{s=j}^{i-1} g_s = g_i \quad \text{if} \quad i > j. \tag{3.2}$$

Upon setting $i = j - 2, j - 3, \ldots, 1$ in succession in (3.1), we find that

$$g_i = \mathbf{0}, \quad i = 1, 2, \ldots, j - 1;$$

upon setting $i = j + 1, j + 2, \ldots$ in (3.2), we have

$$g_i = \mathbf{0}, \quad i = j + 1, j + 2, \ldots .$$

Thus $G = \underset{\sim}{\mathbf{0}}$.

## 4. DECIMATED SEQUENCE, COMPLEMENTARY SEQUENCE, AND SUBMATRIX SEQUENCE OF A BMS

*Theorem 4*: If $G$ is a BMS of eventual period $P$ from $j_0$ and $G^*$ a sequence of submatrices of $G$, then $G^*$ is of eventual period $p$ from $i_0$ where $p|P$ and $i_0 \le j_0$.

*Proof*: If $G$ is eventually periodic from $j_0$, then $G^*$ should also be eventually periodic from $j_0$. As the converse is not true, $G^*$ could have eventual period $p$, where $p|P$, from $i_0 \le j_0$.

*Example*: Consider

$$G = \left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \ldots\right)$$

Here $G$ has eventual period 2 from $i_0 = 3$.

$$G^* = \left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \ldots\right)$$

obtained by taking first and second rows has eventual period 1 from $j_0 = 2$.

*Corollary*: If a BMS $G$ has eventual period $p$ from $j_0$ where $p$ is a prime, then $G^*$ has eventual period $p$ or 1 from $i_0 \le j_0$.

*Theorem 5*: If $G = \left(g_i\right)_{i=1}^{\infty}$ is a BMS of eventual period $p$ from $i_0$, then its decimated sequence $G^d = \left(g_{kd}\right)_{k=1}^{\infty}$ has eventual period $\frac{p}{(p,d)}$ from $k_0 \le \left[\frac{i_0}{d}\right]+1$ where $\left[\frac{i_0}{d}\right]$ is the integral part of $\frac{i_0}{d}$.

*Proof*: Consider $G^d = \left(g_{kd}\right)_{k=1}^{\infty}$. Here

$$g_i = g_{kd} = g_{(k+l)d} \quad \text{for} \quad i \ge i_0,$$

if

$$(k + l)d = i + mp,$$

where $l$, $k$, $m$ are positive integers. Therefore, $G^d$ is periodic for $kd \ge i_0$ if $ld = mp$; i.e., $l = mp/d$. The lowest form of $l = p/(p,d)$. Now, consider the case $d < i_0$. The $n$-tuple $(g_d, g_{2d}, \ldots, g_{nd})$, where $nd < i_0 \le (n + 1)d$ may include a part of the periodic cycle

$$\left(g_{(n+1)d}, g_{(n+2)d}, \ldots, g_{(n+l)d}\right)$$

followed by some full cycles or vice versa. Hence, $G^d$ is of eventual period $l = \frac{p}{(p,n)}$ from $k_0 \le \left[\frac{i_0}{d}\right] + 1$.

*Theorem 6*: If $\overline{G}$ is the complement of the BMS $G$, $D^nG = D^n\overline{G}$.

*Proof*: As $G + \overline{G} = \underset{\sim}{\mathbf{g}}$, $D^nG + D^n\overline{G} = D^n\underset{\sim}{\mathbf{g}} = \underset{\sim}{\mathbf{0}}$. Hence, $D^nG = D^n\overline{G}$.

## 5. AN APPLICATION OF THE BINARY SEQUENCE

Here we describe the binary sequence method to show that $\binom{p+r}{r}$ is odd and $\binom{p+r+1}{r}$ is alternately odd and even for $r = 0, 1, 2, \ldots, n - 1$ when $p = 2^m q$ and $2^{m-1} < n \le 2^m$. Let $\left(a_i\right)_{i=1}^{n}$ be a sequence of integers, and let $H = \left(h_i\right)_{i=1}^{\infty}$ be an infinite sequence where $h_i = a_i$, if $i \le n$, and $h_i = 0$ if $i > n$. Now we construct the binary sequence $G = \left(g_i\right)_{i=1}^{\infty}$ where $g_i$ = the binary 1 or 0, according as $h_i$, i.e., $a_i$, is odd or even. As far as the odd or even nature of the numbers is considered, we are fully justified in the representation of $H$ by the binary sequence $G$, because the integers strictly obey the laws of addition modulo 2, viz.,

(i) the sum of two odd (or even) numbers is even, i.e.,

$$1 + 1 = 0 + 0 = 0,$$

(ii) the sum of an odd number and an even number is odd, i.e.,

$$1 + 0 = 0 + 1 = 1.$$

Now we prove the following theorem.

*Theorem 7*: If $m$ is a positive integer and $p = 2^m q$, where $q$ is odd, then

(7.1) (i) $\binom{p+r}{r}$, $r = 0, 1, 2, \ldots, n - 1$ are all odd,

and

(7.2) (ii) $\binom{p+r+1}{r}$, $r = 0, 1, 2, \ldots, n-1$ are alternately odd and even, where $2^{m-1} < n \le 2^m$.

*Proof*: We represent the sequence of the natural numbers up to $n$ in reverse order in the infinite sequence form

$$H = \left(h_i\right)_{i=1}^{\infty}, \quad \text{where } h_i = n + 1 - i \text{ if } i \le n \text{ and } h_i = 0 \text{ if } i > n. \tag{7.3}$$

Now we represent $H$ by the binary sequence

$$G = (101010\ldots101000\ldots) \text{ or } (010101\ldots01000\ldots)$$

according as $n$ is odd or even, where the binary 1 indicates that the corresponding entry $h_i$ in $H$ is odd, the last appearing binary 1 being the $n$th entry in $G$.

It is evident that $D^pG \equiv D^pH$ and $I_j^pG \equiv I_j^pH$.* In the usual notation

$$I_{n+1}^pH = \left(h_{i,p}\right)_{i=1}^{\infty}, \quad \text{where } h_{i,p} = \begin{cases} \binom{n-i+p+1}{n-i} & \text{if } i \le n \\ 0 & \text{if } i > n. \end{cases}$$

The corresponding binary sequence $I_{n+1}^pG$ is identical with $G$ if

$$\binom{p+1}{0}, \binom{p+2}{1}, \ldots, \binom{p+n}{n-1}$$

are alternately odd and even.

We recall that $DI_jG = G$ [2]. Therefore,

$$DI_{n+1}^pG = DI_{n+1}(I_{n+1}G) = I_{n+1}^pG.$$

Now we differentiate $I_{n+1}^pG$ and find that

$$I_{n+1}^{p-1}G = (111\ldots1000\ldots)$$

in order that $I_{n+1}^pG = G$, the last appearing binary 1 being the $n$th entry. This represents $I_{n+1}^{p-1}H = \left(h_{i,p-1}\right)_{i=1}^{\infty}$, where

$$h_{i,p-1} = \begin{cases} \binom{n-i+p}{n-i} & \text{if } i \le n \\ 0 & \text{if } i > n. \end{cases}$$

It follows that $\binom{p+r}{r}$, $r = 0, 1, 2, \ldots, n-1$ are all odd. Similarly, we find that

$$I_{n+1}^{p-2}G = (00\ldots01000\ldots)$$

representing $I_{n+1}^{p-2}H$, and we have that $\binom{p+r-1}{r}$, $r = 1, 2, \ldots, n-1$ are all

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*Here the definitions $I_j^pG$ and $D^pG$ for a BMS are extended to any eventually null infinite sequence of numbers.*

even. This is true when $p = 2^m q$ where $m$ is a positive integer and $q$ is odd and $2^{m-1} < n \le 2^m$. Now we conclude that $\binom{p+r}{r}$ is odd and $\binom{p+r+1}{r}$ is alternately odd and even for $r = 0, 1, 2, \ldots, n-1$ where $p = 2^m q$ and $2^{m-1} < n \le 2^m$.

*Remark 1*: Care must be taken not to apply the results of Theorem 2 directly in order to obtain the results of Theorem 5. Similarly, the properties of the derivatives and the integrals of a BMS should not be applied directly to $H$ in (7.3).

*Remark 2*: The authors earnestly hope that the reader will be able to find further applications of the binary sequences of BMSs.

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# RESTRICTED MULTIPARTITE COMPOSITIONS

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## 1. INTRODUCTION

In [1] the writer discussed the number of compositions

(1.1) $$n = a_1 + a_2 + \cdots + a_k$$

in positive (or nonnegative) integers subject to the restriction

(1.2) $$a_i \ne a_{i+1} \quad (i = 1, 2, \ldots, k-1).$$

In [2] he considered the number of compositions (1.1) in nonnegative integers such that

(1.3) $$a_i \not\equiv a_{i+1} \pmod{m} \quad (i = 1, 2, \ldots, k-1),$$

where $m$ is a fixed positive integer.

In the present paper we consider the number of *multipartite compositions*

(1.4) $$n_j = a_{j1} + a_{j2} + \cdots + a_{jk} \quad (j = 1, 2, \ldots, t)$$

in nonnegative $a_{js}$ subject to

(1.5) $$\mathbf{a}_i \ne \mathbf{a}_{i+1} \quad (i = 1, 2, \ldots, k-1)$$

or

(1.6) $$\mathbf{a}_i \not\equiv \mathbf{a}_{i+1} \pmod{m} \quad (i = 1, 2, \ldots, k-1)$$