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THE STUDY OF POSITIVE INTEGERS (a,b) SUCH THAT $ab+1$ IS A SQUARE

PETER HEICHELHEIM
Peat Marwick and Partners, Toronto, Ontario, Canada

1. INTRODUCTION

A P-set will be defined as a set of positive integers such that if a and b are two distinct elements of this set, $ab+1$ is a square.

There are many examples of P-sets such as [2, 12] or [1, 3, 8, 120] and even formulas such as

$$[n-1,\ n+1,\ 4n,\ 4n(4n^2-1)]$$

or

$$[m,\ n^2-1+(m-1)(n-1)^2,\ n(mn+2),\ 4m(mn^2-mn+2n-1)^2 + 4(mn^2-mn+2n-1)].$$

(See Cross [1].) However, none of these formulas are general.

More recently, there has been considerable work on P-sets with polynomials (by Jones [2, 3]) and in connection with Fibonacci numbers (by Hoggatt and Bergum [4]).

It is of interest to find out how much these sets can be extended by adding new positive integers to the set; for example [2, 12] can be extended to [2, 12, 420]. A P-set which cannot be extended will be called nonextendible. One purpose of this article is to show that a nonextendible set must have at least four members. Then it will be demonstrated that the number of members of a P-set is finite. Finally, it will be shown that certain types of five-member P-sets will be impossible.

2. EXTENDING P-SETS TO FOUR ELEMENTS

The proof that sets of one or two elements are extendible is very simple, for $[N]$ can always be extended to $[N, N+2]$ and $[a, b]$ can be extended to $[a, b, a+b+2x]$ where $x^2 = ab+1$. (See Euler [5].)

Let $[a, b, N]$ be members of a P-set. Then,

(1) $ab + 1 = x^2$,

(2) $aN + 1 = y^2$,

(3) $bN + 1 = z^2$.

Therefore,

(4) $$by^2 - az^2 = b - a.$$

Let $\overline{y} = by$. Then,

(5) $$\overline{y} - abz^2 = b(b - a).$$

Let the auxiliary Pell equation of (5) be

(6) $$m^2 - abn^2 = 1$$

or

(7) $$m^2 - (x^2 - 1)n^2 = 1.$$

The minimal positive solution of (7) is $(x,1)$. Hence all the solutions of (7) are given by

$$m_i + \sqrt{x^2 - 1}n_i = (x + \sqrt{x^2 - 1})^i, \quad i = 1, 2, 3, \ldots,$$

and all solutions of (5) are given by

(8) $$\overline{y}_i + \sqrt{x^2 - 1}z_i = (\overline{y}_0 + \sqrt{x^2 - 1}z_0)(x + \sqrt{x^2 - 1})^i, \quad i = 0, 1, 2, \ldots,$$

where $(\overline{y}_0, z_0)$ can take only a finite number of values, one of which must be $(b,1)$. (See Nagell [6].)

There is a one-to-one correspondence between the solutions (y_i, z_i) of (4) and $(\overline{y}_i, z_i)$ of (5) where $\overline{y}_i = by_i$ because $\overline{y}_i^2 = b(b - a + az_i^2)$, and hence y_i is always an integer.

Theorem 1: Let

(9) $$N_k = \frac{y_k^2 - 1}{a}.$$

Then

(10) $$N_i N_{i+j} + 1 = (m_j N_i + n_j y_i z_i)^2 + 1 - n_j^2.$$

Proof: From (8),

$$\overline{y}_{i+j} + \sqrt{x^2 - 1}z_{i+j} = (\overline{y}_i + \sqrt{x^2 - 1}z_i)(m_j + \sqrt{x^2 - 1}n_j).$$

Then $\overline{y}_{i+j} = m_j\overline{y}_i + n_j(x^2 - 1)z_i.$

Hence $by_{i+j} = bm_j y_i + abn_j z_i.$

Therefore $y_{i+j} = m_j y_i + an_j z_i.$

Hence $N_{i+j} = \frac{1}{a}(m_j^2 y_i^2 + 2am_j n_j y_i z_i + a^2 n_j^2 z_i^2 - 1)$, using (9).

Therefore
$$\begin{aligned} N_i N_{i+j} + 1 &= \frac{1}{a^2}(y_i^2 - 1)(m_j^2 y_i^2 + 2am_j n_j y_i z_i + a^2 n_j^2 z_i^2 - 1) + 1 \\ &= \frac{1}{a^2}(m_j^2 y_i^4 + 2am_j n_j z_i y_i^3 + [a^2 n_j^2 z_i^2 - 1 - m_j^2]y_i^2 \\ &\quad - 2am_j n_j z_i y_i - an_j^2[by_i^2 - b + a] + 1 + a^2), \\ &\qquad \text{using (4)}, \\ &= \frac{1}{a^2}(m_j^2 y_i^4 + 2am\, n_j z_i y_i^3 + [a^2 n_j^2 z_i^2 - 1 - m_j^2 - abn_j^2]y_i^2 \end{aligned}$$

(continued)

$$- 2am_j n_j z_i y_i + abn_j^2 - a^2 n_j^2 + 1 + a^2)$$

$$= \frac{1}{a^2}(m_j^2 y_i^4 + 2am_j n_j z_i y_i^3 + [a^2 n_j^2 z_i^2 - 2m_j^2] y_i^2$$

$$- 2am_j n_j z_i y_i + m_j^2) + 1 - n_j^2, \text{ using (6)},$$

$$= (m_j N_i + n_j y_i z_i)^2 + 1 - n_j^2, \text{ using (9)}.$$

Theorem 2: The P-set $[a, b, N_i]$ can be extended to $[a, b, N_i, N_{i+1}]$.

Proof: Now $y_{i+1} = m_1 y_i + an_1 z_i > y_i$.

Therefore $N_{i+1} > N_i$, using (9).

Therefore N_{i+1} is positive if N_i is positive.

Also, if N_i is an integer, $y_i^2 \equiv 1 \bmod a$. Now,

$$\begin{aligned} y_{i+1}^2 &= m_1^2 y_i^2 + 2am_1 n_1 y_i z_i + a^2 n_1^2 z_i^2 \\ &\equiv (ab + 1) y_i^2 \bmod a \text{ as } m_1 = x = \sqrt{ab + 1} \\ &\equiv y_i^2 \bmod a. \end{aligned}$$

Therefore N_{i+1} is an integer.

In fact, it can be shown by induction that if N_i is a positive integer, then so must be N_{i+j}.

Now as $(m_1, n_1) = (x, 1)$, then

$$N_i N_{i+1} + 1 = (xN_i + y_i z_i)^2$$

and therefore $[a, b, N_i]$ can be extended to $[a, b, N_i, N_{i+1}]$.

A formula can be developed for N_{i+1} from a, b, and N_i; that is,

$$N_{i+1} = a + b + N_i + 2abN_i + 2\sqrt{(ab + 1)(aN_i + 1)(bN_i + 1)}.$$

3. FINITENESS OF P-SETS

There are no known P-sets of more than four members. However, it can be proved that there are no infinite sets. In fact, given three members of the set a, b, and c, it can be shown that all other members are bounded, for if

$$aN + 1 = x^2,\ bN + 1 = y^2,\ cN + 1 = z^2, \text{ and } t = xyz,$$

then $abcN^3 + (ab + bc + ca)N^2 + (a + b + c)N + 1 = t^2$.

Let $H = \max\{abc, ab + bc + ca, a + b + c\}$.

Now, as $abcN^3 + (ab + bc + ca)N^2 + (a + b + c)N + 1$ has no squared linear factor in N, by Baker [7],

$$N < \exp\{(10^6 H)^{10^6}\}.$$

Until recently there was no way of knowing if a P-set was nonextendible if it had four elements. However, in Baker and Davenport [8], it has been proved that [1, 3, 8, 120] cannot be extended. In fact, it has been shown that [1, 3, 8] can only be extended to [1, 3, 8, 120]. There were calculations done to prove this that needed the aid of a computer. The method in Baker and Davenport [8] would seem workable for checking if there are other sets of four which are nonextendible.

A recent adaptation of this method was given by Grinstead [9].

4. RESTRICTIONS ON EXTENDING FOUR-MEMBER P-SETS

First it should be noted that Baker and Davenport [8] (from their relationship (20) and Section 5) seem to indicate that any fifth member of a P-set that is very large compared to the first four would have to satisfy some very unusual conditions.

The following lemma and theorem give some limitations in the reverse direction.

Lemma: $x < a + b$ if $a > 0$ and $b > 0$.

Proof: If $x \geq a + b$, then $a^2 + ab + b \leq 1$, using (1). Therefore, $a = 0$ or $b = 0$, which is not true.

Theorem 3: If $2i \geq j - 1$, then with the exception of $j = 1$, $N_i N_{i+1} + 1$ is not a square. (N_i is not equal to zero, as members of P-sets are defined to be positive.)

Proof: Let $L = m_j N_i + n_j y_i z_i$. Suppose $N_i N_{i+j} + 1$ is a square. Now

$$L^2 - (L - 1)^2 = 2L - 1.$$

Then

$$n_j^2 - 1 \geq 2L - 1 \quad \text{from (10)} \quad \text{if } j \neq 1.$$

Therefore

$$\frac{m_j N_i + n_j y_i z_i}{\frac{n_j^2}{2}} \leq 1. \tag{11}$$

Now

$$by_i + \sqrt{ab}z_i = (by_0 + \sqrt{ab}z_0)(m_i + \sqrt{ab}n_i)$$

implies

$$y_i = m_i y_0 + a n_i z_0$$

and

$$z_i = b n_i y_0 + m_i z_0.$$

Let $M = x + \sqrt{x^2 - 1}$. Then it can be shown that

$$m_j = \frac{1}{2}(M^j + M^{-j})$$

and

$$n_j = \frac{1}{2\sqrt{x^2 - 1}}(M^j - M^{-j}).$$

Therefore

$$\frac{n_j^2}{2} = \frac{1}{8(x^2 - 1)}(M^j - M^{-j})^2.$$

Now

$$y_i \geq \frac{1}{2}(M^i + M^{-i}) + \frac{a}{2\sqrt{x^2 - 1}}(M^i - M^{-i})$$

and

$$z_i \geq \frac{b}{2\sqrt{x^2 - 1}}(M^i - M^{-i}) + \frac{1}{2}(M^i + M^{-i}).$$

Then

$$y_i z_i \geq \frac{b}{4\sqrt{x^2 - 1}}(M^{2i} - M^{-2i}) + \frac{ab}{4(x^2 - 1)}(M^i - M^{-i})^2$$

$$+ \frac{1}{4}(M^i + M^{-i})^2 + \frac{a}{4\sqrt{x^2 - 1}}(M^{2i} - M^{-2i})$$

$$= \frac{a + b}{4\sqrt{x^2 - 1}}(M^{2i} - M^{-2i}) + \frac{1}{2}(M^{2i} + M^{-2i}).$$

Therefore

$$\frac{n_j y_i z_i}{\frac{n_j^2}{2}} \geq \frac{\frac{a + b}{8(x^2 - 1)}(M^{2i} - M^{-2i}) + \frac{1}{4\sqrt{x^2 - 1}}(M^{2i} + M^{-2i})}{\frac{1}{8(x^2 - 1)}(M^j - M^{-j})}$$

$$= (a + b)\frac{(M^{2i} - M^{-2i})}{(M^j - M^{-j})} + 2\sqrt{x^2 - 1}\frac{(M^2 + M^{-2})}{(M^j - M^{-j})} > x\frac{(M^{2i} - M^{-2i})}{(M^j - M^{-j})}$$

$$+ 2\sqrt{x^2 - 1}\frac{(M^{2i} - M^{-2i})}{(M^j - M^{-j})}$$

(as $x < a + b$ from the Lemma)

$$= \frac{M^{2i+1} - M^{-2i+1} + M^{2i}\sqrt{x^2 - 1} - M^{-2i}\sqrt{x^2 - 1}}{M^j - M^{-j}}.$$

Now, if $i > 0$, it can be easily shown that

$$M^{-4i} + \frac{M^{-4i+1}}{\sqrt{x^2 - 1}} < 1 \quad \text{as} \quad x > 1.$$

Therefore $\quad -M^{-2i+1} + M^{2i}\sqrt{x^2 - 1} - M^{-2i}\sqrt{x^2 - 1} > 0 > -M^{-2i-1}.$

Hence

$$\frac{n_j y_i z_i}{\frac{n_j^2}{2}} > \frac{M^{2i+1} - M^{-(2i+1)}}{M^j - M^{-j}} \geq 1 \quad \text{if} \quad 2i + 1 \geq j \text{ or } j - 1 \leq 2i.$$

Thus, Theorem 3 is proved, from (11), as $j > 1$; therefore, i must be greater than zero.

5. A PARTICULAR RATIONAL FIVE-MEMBER P-SET BY EULER CANNOT BE INTEGER

It will now be shown what will happen if rationals are allowed. Suppose the P-set $[a, b, c, d]$ is extended to

$$\left[a, b, c, d, \frac{4r + 2p(s + 1)}{(s - 1)^2}\right]$$

where a, b, c, and d are positive integers,

$$c = 2x + a + b,\ d = 4x(x + a)(x + b),$$

$$p = a + b + c + d,\ r = abc + abd + acd + bcd,$$

and $$s = abcd.$$

Here, s can have a positive or negative value. This was first given by Euler [5].

Theorem 4: $\dfrac{4r + 2p(s + 1)}{(s - 1)^2}$ is never a positive integer. In fact, it is always less than 1.

The following proof has been considerably shortened due to some suggestions from Professor Jones.

Proof: Re-order a, b, c, and d such that $a < b < c < d$. If $a = 1$ and $b = 2$, then $ab + 1 = 3$, which is not a square. Therefore $b \geq 3$, $c \geq 4$, and $d \geq 5$. Now

$$\frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd} \leq \frac{13}{60} < \frac{1}{4}.$$

Therefore $4p < s$. Also,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < 2.$$

Therefore $abc + abd + acd + bcd < 2abcd$ or $r < 2s$.

$$\text{Hence } \frac{4r + 2p(s + 1)}{(s - 1)^2} < \frac{8s + \dfrac{s(s + 1)}{2}}{(s - 1)^2} = \frac{s^2 + 17s}{2(s - 1)^2}$$

$$= \frac{1}{2} + \frac{19}{2(s - 1)} + \frac{9}{(s - 1)^2}$$

$$< \frac{1}{2} + \frac{19}{116} + \frac{9}{3364} \quad \text{as} \quad s > 59$$

$$< 1.$$

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