ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months of publication of the problems.

H-307 *Proposed by Larry Taylor, Briarwood, NY*

(A) If $p \equiv \pm 1 \pmod{10}$ is prime, $x \equiv \sqrt{5}$, and $a \equiv \dfrac{2(x-5)}{x+7} \pmod{p}$, prove that a, $a+1$, $a+2$, $a+3$, and $a+4$ have the same quadratic character modulo p if and only if $11 < p \equiv 1$ or $11 \pmod{60}$ and $(-2x/p) = 1$.

(B) If $p \equiv 1 \pmod{60}$, $(2x/p) = 1$, and $b \equiv \dfrac{-2(x+5)}{7-x} \pmod{p}$, then b, $b+2$, $b+3$, and $b+4$ have the same quadratic character modulo p. Prove that $(11ab/p) = 1$.

H-308 *Proposed by Paul Bruckman, Concord, CA*

Let $[a_1, a_2, \ldots, a_n] = \dfrac{p_n}{q_n} \quad \dfrac{p_n(a_1, a_2, \ldots, a_n)}{q_n(a_1, a_2, \ldots, a_n)}$ denote the nth convergent of the infinite simple continued fraction $[a_1, a_2, \ldots]$, $n = 1, 2, \ldots$. Also, define $p_0 = 1$, $q_0 = 0$. Further, define

$$\begin{aligned} W_{n,k} &= p_n(a_1, a_2, \ldots, a_n)q_k(a_1, a_2, \ldots, a_k) \\ &\quad - p_k(a_1, a_2, \ldots, a_k)q_n(a_1, a_2, \ldots, a_n) \\ &\quad p_n q_k - p_k q_n, \quad 0 \le k \le n. \end{aligned} \tag{1}$$

Find a general formula for $W_{n,k}$.

H-309 *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

Let f be a permutation of $\{1, 2, \ldots, m-1\}$ such that the terms $i + f(i)$ are all distinct (mod m). Characterize and/or enumerate such f. [Each such f gives a decomposition of the $m(m+1)$ m-nomial coefficients, which are the nearest neighbors of a given m-nomial coefficient, into m sets of $m+1$ coefficients which have equal products and are congruent by rotation—see Hoggatt & Alexanderson, "A Property of Multinomial Coefficients," *The Fibonacci Quarterly* 9, No. 4 (1971):351-356, 420-421.]

I have run a simple program to generate and enumerate such f, but can see no pattern. The number N of such permutations is given below for $m \le 10$. The ratio $N/(m-1)!$ is decreasing steadily leading to the conjecture that it converges to 0.

m	2	3	4	5	6	7	8	9	10
N	1	1	2	3	8	19	64	225	928

H-310 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $\alpha = (1+\sqrt{5})/2$, $[n\alpha] = a_n$, and $[n\alpha^2] = b_n$. Clearly, $a_n + n = b_n$.

a) Show that if $n = F_{2m+1}$, then $a_n = F_{2m+2}$ and $b_n = F_{2m+3}$.

b) Show that if $n = F_{2m}$, then $a_n = F_{2m+1} - 1$ and $b_n = F_{2m+2} - 1$.

c) Show that if $n = L_{2m}$, then $a_n = L_{2m+1}$ and $b_n = L_{2m+2}$.

d) Show that if $n = L_{2m+1}$, then $a_n = L_{2m+2} - 1$ and $b_n = L_{2m+3} - 1$.

SOLUTIONS

Editorial Note: *Starting with this issue, we shall indicate the issue and date when each problem was proposed.*

Continue

H-278 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA (Vol. 16, No. 1, Feb. 1978)*

Show $$\sqrt{\frac{5F_{n+2}}{F_n}} = \langle 3, \overline{\underbrace{1, 1, \ldots, 1}_{n-1}, 6}\rangle$$

(Continued fraction notation, cyclic part under bar.)

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA

$$D = \frac{F_{n+2}}{F_n} = L_2 \quad \text{and remainder} \quad (-\alpha^{n-2} - \beta^{n-2})$$

$$2 \le D \le 3$$

$$10 \le 5D \le 15$$

$$[\sqrt{5D}] = 3.$$

$(F_n, F_{n+1}) = 1$ implies $(F_n, F_{n+2}) = 1$. $\sqrt{5D}$ has a unique periodic C.F. expansion with first element 3 and terminal element 6.

$$L_{n+1}^2 - \frac{5F_{n+2}}{F_n}F_n^2 = L_{2n+2} + 2(-1)^{n+1} - L_{2n+2} + (-1)^n L_2 = (-1)^n,$$

$$x = L_{n+1},\ y = F_n$$

is a solution of $x^2 - 5Dy^2 = \pm 1$.

For the p_i and q_i convergents formed from the C.F. expansion of $\sqrt{5D}$ to terminate with $p_n = L_{n+1}$ and $q_n = F_n$, the middle elements must be $(n-1)$ ones.

Also solved by the proposer.

A Rare Mixture

H-279 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA*
(Vol. 16, No. 1, Feb. 1978)

Establish the F-L identities:

(a) $F^4_{n+6r} - (L_{4r} + 1)(F^4_{n+4r} - F^4_{n+2r}) - F^4 = F_{2r}F_{4r}F_{6r}F_{4n+12r}$

(b) $F^4_{n+6r+3} + (L_{4r+2} - 1)(F^4_{n+4r+2} - F^4_{n+2r+1}) - F^4_n$
$= F_{2r+1}F_{4r+2}F_{6r+3}F_{4n+12r+6}$.

Solution by Paul Bruckman, Concord, CA

Lemma 1: $L_{3m} - (-1)^m L_m = 5F_m F_{2m}$.

Proof: $L_{3m} - (-1)^m L_m = a^{3m} + b^{3m} - (ab)^m(a^m + b^m)$
$= (a^m - b^m)(a^{2m} - b^{2m}) = 5F_m F_{2m}$.

Lemma 2: $5(F^4_u - F^4_v) = F_{u-v}F_{u+v}(L_{u-v}L_{u+v} - 4(-1)^u)$.

Proof: $25F^4_u = (a^u - b^u)^4 = a^{4u} + b^{4u} - 4(-1)^u(a^{2u} + b^{2u}) + 6$.

Therefore,

$$\begin{aligned}
25(F^4_u - F^4_v) &= a^{4u} - a^{4v} + b^{4u} - b^{4v} - 4(-1)^u a^{2u} + 4(-1)\ a^2 \\
&\qquad - 4(-1)^u b^{2u} + 4(-1)^v b^{2v} \\
&= (a^{2u+2v} - b^{2u+2v})(a^{2u-2v} - b^{2u-2v}) \\
&\qquad - 4(-1)^u a^{u+v}(a^{u-v} - (-1)^{v-u}a^{v-u}) \\
&\qquad - 4(-1)^u b^{u+v}(b^{u-v} - (-1)^{v-u}b^{v-u}) \\
&= 5F_{2u+2v}F_{2u-2v} - 4(-1)^u(a^{u+v} - b^{u+v})(a^{u-v} - b^{u-v}) \\
&= 5F_{2u+2v}F_{2u-2v} - 20(-1)^u F_{u+v}F_{u-v} \\
&= 5F_{u+v}F_{u-v}(L_{u+v}L_{u-v} - 4(-1)^u),
\end{aligned}$$

which implies the statement of the lemma.

Lemma 3: $(-1)^m L_{2m} + 1 = (-1)^m F_{3m}/F_m$.

Proof: $(-1)^m L_{2m} + 1 = (-1)^m(L_{2m} + (-1)^m) = (-1)^m(a^{2m} + a^m b^m + b^{2m})$

$$= (-1)^m\left\{\frac{a^{3m} - b^{3m}}{a^m - b^m}\right\} = (-1)^m\ \frac{F_{3m}}{F_m}.$$

Now

$$\begin{aligned}
&F^4_{n+3m} - ((-1)^m L_{2m} + 1)(F^4_{n+2m} - F^4_{n+m}) - F^4_n \\
&= \tfrac{1}{5}F_{3m}F_{2n+3m}(L_{3m}L_{2n+3m} - 4(-1)^{n+3m}) \\
&\qquad - ((-1)^m L_{2m} + 1)\tfrac{1}{5}F_m F_{2n+3m}\ (L_m L_{2n+3m} - 4(-1)^{n+2m})
\end{aligned}$$

(applying Lemma 2 twice, with $u = n + 3m$, $v = n$ and $u = n + 2m$, $v = n + m$)

$$\begin{aligned}
&= \frac{1}{5}F_{2n+3m}L_{2n+3m}\ F_{3m}L_{3m} - ((-1)^m L_{2m} + 1)F_m L_m \\
&\qquad - \frac{4}{5}(-1)^{n+2m}F_{2n+3m}\Big((-1)^m F_{3m} - ((-1)^m L_{2m} + 1)F_m\Big)
\end{aligned}$$

$$= \frac{1}{5} F_{2n+3m} L_{2n+3m} \left(F_{3m} L_{3m} - (-1)^m F_{3m} L_m\right)$$
$$- \frac{4}{5}(-1)^n F_{2n+3m} \left((-1)^m F_{3m} - (-1)^m F_{3m}\right)$$

(applying Lemma 3)

$$= \frac{1}{5} F_{3m} F_{4n+6m} \left(L_{3m} - (-1)^m L_m\right)$$

$$= F_m F_{2m} F_{3m} F_{4n+6m} \text{ (by Lemma 1).}$$

Therefore:

$$F^4_{n+3m} - \left((-1)^m L_{2m} + 1\right)\left(F^4_{n+2m} - F^4_{n+m}\right) - F^4 = F_m F_{2m} F_{3m} F_{4n+6m}.$$

Setting $m = 2r$ and $m = 2r + 1$ yields (a) and (b), respectively.

Also solved by the proposer.

Mod Ern

H-280 *Proposed by P. Bruckman, Concord, CA (Vol. 16, No. 1, Feb. 1978)*

Prove the congruences

(1) $F_{3\cdot 2^n} \equiv 2^{n+2} \pmod{2^{n+3}}$;

(2) $L_{3\cdot 2^n} \equiv 2 + 2^{2n+2} \pmod{2^{2n+4}}$, $n = 1, 2, 3, \ldots$.

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA

(1) $n = 1$, $F_6 = 8 \equiv 8 \pmod{16}$.

Since $L^2_{3k} - 5F^2_{3k} = \pm 4$, $(L_{3k}, F_{3k}) = 2$,

$2^r \mid F_{3\cdot 2^n}$, $r > 1$, $n \geq 1$;

$2^t \mid L_{3\cdot 2^n}$ if and only if $t = 1$.

Assume $F_{3\cdot 2^n} \equiv 2^{n+2} \pmod{2^{3}} \equiv 2^{n+2} \pmod{2^{n+4}}$

$F_{3\cdot 2^{n+1}} = F_{3\cdot 2^n} L_{3\cdot 2^n} \equiv 2^{n+3} \pmod{2^{n+4}}$.

(2) $n = 1$, $L_6 = 18 \equiv 2 + 2^4 \pmod{2^6}$, $L^2_{2k} = L_{4k} + 2$.

Assume $L_{3\cdot 2^n} \equiv 2 + 2^{2n+2} \pmod{2^{2n+4}}$

$L_{3\cdot 2^{n+1}} = L^2_{3\cdot 2^n} - 2 \equiv 2 + 2^{2n+4} + 2^{4n+4} \pmod{2^{4n+5}}$

$L_{3\cdot 2^{n+1}} \equiv 2 + 2^{2n+4} \pmod{2^{2n+6}}$, $n \geq 1$.

Also solved by the proposer, who noted that this is Corollary 6 in "Periodic Continued Fraction Representations of Fibonacci-type Irrationals," by V. E. Hoggatt, Jr. & Paul S. Bruckman, in The Fibonacci Quarterly 15, No. 3 (1977): 225-230.

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