ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR. S.E., ALBUQUERQUE, NEW MEXICO 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-412 *Proposed by Phil Mana, Albuquerque, NM*

Find the least common multiple of the integers in the infinite set

$$\{2^9 - 2,\ 3^9 - 3,\ 4^9 - 4,\ \ldots,\ n^9 - n,\ \ldots\}.$$

B-413 *Proposed by Herta T. Freitag, Roanoke, VA*

For every positive integer n, let U_n consist of the points $j + ke^{2\pi i/3}$ in the Argand plane with $j \in \{0, 1, 2, \ldots, n\}$ and $k \in \{0, 1, \ldots, j\}$. Let $T(n)$ be the number of equilateral triangles whose vertices are subsets of U_n. For example, $T(1) = 1$, $T(2) = 5$, and $T(3) = 13$.

a. Obtain a formula for $T(n)$;
b. Find all n for which $T(n)$ is an integral multiple of $2n + 1$.

B-414 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $S_n = L_{n+5} + \binom{n}{2}L_{n+2} - \sum_{i=2}^{n}\binom{i}{2}L_i - 11$. Determine all n in $\{2, 3, 4, \ldots\}$ for which S_n is (a) prime; (b) odd.

B-415 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

The circumference of a circle in a fixed plane is partitioned into n arcs of equal length. In how many ways can one color these arcs if each arc must be red, white, or blue? Colorings which can be rotated into one another should be considered to be the same.

B-416 *Proposed by Gene Jakubowski and V. E. Hoggatt, Jr.*
San Jose State University, San Jose, CA

Let F_n be defined for all integers (positive, negative, and zero) by

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n,$$

and hence

$$F_n = F_{n+2} - F_{n+1}.$$

Prove that every positive integer m has at least one representation of the form

$$m = \sum_{-N}^{N} \alpha_j F_j,$$

with each α_j in $\{0, 1\}$ and $\alpha_j = 0$ when j is an integral multiple of 3.

B-417 *Proposed by R. M. Grassl and P. L. Mana*
University of New Mexico, Albuquerque, NM

Here let $[x]$ be the greatest integer in x. Also, let $f(n)$ be defined by $f(0) = 1 = f(1)$, $f(2) = 2$, $f(3) = 3$, and

$$f(n) = f(n-4) + [1 + (n/2) + (n^2/12)]$$

for $n \in \{4, 5, 6, \ldots\}$. Do there exist rational numbers a, b, c, d such that

$$f(n) = [a + bn + cn^2 + dn^3]?$$

SOLUTIONS

Partitioning Squares Near the Diagonals

B-388 *Proposed by Herta T. Freitag, Roanoke, VA*

Let T_n be the triangular number $n(n+1)/2$. Show that

$$T_1 + T_2 + T_3 + \cdots + T_{2n-1} = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$$

and express these equal sums as a binomial coefficient.

Solution by Phil Mana, Albuquerque, NM

It is readily seen that $T_1 = 1 = 1^2$ and $T_{2k} + T_{2k+1} = (2k+1)^2$ for $k = 1, 2, \ldots$. The displayed equation then follows. Next one notes that

$$\begin{aligned} T_1 + T_2 + \cdots + T_{2n-1} &= \binom{2}{2} + \binom{3}{2} + \cdots + \binom{2n}{2} \\ &= \binom{3}{3} + \left[\binom{4}{3} - \binom{3}{3}\right] + \cdots + \left[\binom{2n+1}{3} - \binom{2n}{3}\right] \\ &= \binom{2n+1}{3}. \end{aligned}$$

Also solved by Paul Bracken, Wray G. Brady, Paul S. Bruckman, R. Garfield, Hans Klauser (Switzerland), Peter A. Lindstrom, Graham Lord, Ellen R. Miller, C. B. A. Peck, Bob Prielipp, A. G. Shannon (Australia), Sahib Singh, Paul Smith, Lawrence Somer, Rolf Sonntag (W. Germany), Gregory Wulczyn, and proposer.

Transformed Arithmetic Progression

B-389 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Find the complete solution, with two arbitrary constants, of the difference equation

$$(n^2 + 3n + 3)U_{n+2} - 2(n^2 + n + 1)U_{n+1} + (n^2 - n + 1)U_n = 0.$$

Solution by Paul S. Bruckman, Concord, CA

Let

(1) $$V_n = (n^2 - n + 1)U_n.$$

Then, $V_{n+1} = (n^2 + n + 1)U_{n+1}$, $V_{n+2} = (n^2 + 3n + 3)U_{n+2}$, and so

(2) $$V_{n+2} - 2V_{n+1} + V_n = 0,$$

or

(3) $$\Delta^2 V_n = 0.$$

It follows that $V_n = an + b$, for some constants a and b. Note that

$$V_0 = b = U_0, \text{ and } V_1 = a + b = U_1.$$

Hence, $b = U_0$ and $a = U_1 - U_0$, which implies

(4) $$U_n = \frac{(U_1 - U_0)n + U_0}{n^2 - n + 1}.$$

Also solved by Wray G. Brady, R. Garfield, C. B. A. Peck, Sahib Singh, Paul Smith, and proposer.

Generating Diagonals of Pascal's Triangle

B-390 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Find, as a rational function of x, the generating function

$$G_k(x) = \binom{k}{k} + \binom{k+1}{k}x + \binom{k+2}{k}x^2 + \cdots + \binom{k+n}{k}x^n + \cdots, \quad |x| < 1.$$

I. *Solution by Ralph Garfield, College of Insurance; Graham Lord, Université Laval; and Paul Smith, University of Victoria (independently).*

$G_k(x)$ is well known to be $(1 - x)^{-k-1}$. (Consider the Taylor series or Newton binomial expansion of this latter function.)

II. *Solution by Wray G. Brady, Slippery Rock State College; Robert M. Giuli, University of California, Santa Cruz, and Herta T. Freitag, Roanoke, VA (independently).*

First we show the identity $G_k(x) = G_{k-1}(x)/(1 - x)$ or

$$G_{k-1}(x) = (1 - x)G_k(x).$$

$$(1 - x)G\ (x) = \binom{k}{k} + \left[\binom{k+1}{k} - \binom{k}{k}\right]x + \left[\binom{k+2}{k} - \binom{k+1}{k}\right]x^2 + \cdots$$

(continued)

$$= \binom{k-1}{k-1} + \binom{k}{k-1}x + \binom{k+1}{k-1}x^2 + \cdots$$
$$= G_{k-1}(x).$$

Now an induction will show $G_k(x) = 1/(1-x)^{k+1}$ since $G_0(x) = 1/(1-x)$.

III. Solution by Paul Bracken (Toronto); Phil Mana; and C. B. A. Peck (independently).

Let $F_k(x) = (1-x)^{-k-1}$. Then one readily sees that

$$F_0(x) = (1-x)^{-1} = G_0(x),\ F_k(0) = 1 = G_k(0),$$

$$dF_k(x)/dx = (k+1)F_{k+1}(x),\ dG_k(x)/dx = (k+1)G_{k+1}(x).$$

Using integration and induction, one establishes that

$$G_k(x) = F_k(x) = (1-x)^{-k-1} \text{ for } k = 0, 1, 2, \ldots.$$

Also solved by Paul S. Bruckman, A. G. Shannon, Sahib Singh, Gregory Wulczyn, and proposer.

Approximations to Root Five

B-391 *Proposed by M. Wachtel, Zurich, Switzerland*

Some of the solutions of $5x^2 + 1 = y^2$ in positive integers x and y are $(x,y) = (4,9)$, $(72,161)$, $(1292,2889)$, $(23184,51841)$, and $(416020,930249)$. Find a recurrence formula for the x_n and y_n of a sequence of solutions (x_n,y_n) and find $\lim_{n\to\infty}(x_{n+1}/x_n)$ in terms of $\alpha = (1+\sqrt{5})/2$.

Solution by Paul S. Bruckman, Concord, California

The Diophantine equation

(1) $$y^2 - 5x^2 = 1$$

is a special case of the general Pell equation: $y^2 - mx^2 = 1$, where m is not a square. From the theory of the Pell equation, it is known that (1) possesses infinitely many solutions, and indeed that all of the solutions (x_n,y_n) in positive integers are given by the relation:

(2) $$y_n + x_n\sqrt{5} = (y_1 + x_1\sqrt{5})^n,\ n = 1, 2, 3, \ldots,$$

where (x_1,y_1) is the minimal solution.

We readily find that $(x_1,y_1) = (4,9)$. Let $A = 9 + 4\sqrt{5}$ and $B = 9 - 4\sqrt{5}$. Note that $A = (2+\sqrt{5})^2 = \alpha^6$ and $B = A^{-1} = b^6$. Since $y_n - x_n\sqrt{5} = B^n$, it follows that $y_n = (A^n + B^n)/2 = (\alpha^{6n} + b^{6n})/2$, and

$$x_n = \frac{1}{2\sqrt{5}}(A^n - B^n) = \frac{\alpha^{6n} - b^{6n}}{2(\alpha - b)},\ \text{or}$$

(3) $$(x_n,y_n) = \left(\frac{1}{2}F_{6n}, \frac{1}{2}L_{6n}\right),\ n = 1, 2, 3, \ldots.$$

Since $(z-A)(z-B) = z^2 - 18z + 1$, it follows that x_n and y_n satisfy the common recursion:

$$z_{n+2} - 18z_{n+1} + z_n = 0. \tag{4}$$

Moreover, $L \equiv \lim\limits_{n\to\infty} (x \quad /x_n) = \lim\limits_{n\to\infty} \left(\dfrac{A^{n+1} - B^{n+1}}{A^n - B^n}\right) = A$, since $A > 1$, $0 < B < 1$, i.e.,

$$L = \alpha^6. \tag{5}$$

Also solved by Wray G. Brady, C. B. A. Peck, A. G. Shannon, Sahib Singh, Paul Smith, and proposer.

Half-Way Application of $(E^2 - E - 1)^2$

B-392 *Proposed by Phil Mana, Albuquerque, NM*

Let $Y_n = (2 + 3n)F_n + (4 + 5n)L_n$. Find constants h and k such that

$$Y_{n+2} - Y_{n+1} - Y_n = hF_n + kL_n.$$

Solution by Graham Lord, Université Laval, Québec

$$\begin{aligned} Y_{n+2} - Y_{n+1} - Y_n &= (2 + 3n + 6)F_{n+2} + (4 + 5n + 10)L_{n+2} - (2 + 3n + 3)F_{n+1} \\ &\quad - (4 + 5n + 5)L_{n+1} - (2 + 3n)F_n - (4 + 5n)L_n \\ &= 6F_{n+2} - 3F_{n+1} + 10L_{n+2} - 5L_{n+1} = 20F_n + 14L_n. \end{aligned}$$

Thus $h = 20$ and $k = 14$.

Also solved by Paul Bracken, Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, John W. Milsom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Sahib Singh, Paul Smith, Rolf Sonntag, Gregory Wulczyn, and proposer.

Triangle of Triangular Factorials

B-393 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

Let $T_n = \binom{n+1}{2}$, $P_0 = 1$, $P_n = T_1T_2 \ldots T_n$ for $n > 0$ and $\begin{bmatrix} n \\ k \end{bmatrix} = P_n/P_kP_{n-k}$ for integers k and n with $0 \le k \le n$. Show that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{n - k + 1}\binom{n}{k}\binom{n+1}{k+1}.$$

Solution by Paul S. Bruckman, Concord, CA

$$P_n = \prod_{k=1}^{n} T_k = \prod_{k=1}^{n} k(k + 1)/2 = 2^{-n}n!(n + 1)! \quad \text{Therefore}$$

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= \frac{P_n}{P_kP_{n-k}} = \frac{n!(n + 1)!2^k2^{n-k}}{2^nk!(k + 1)!(n - k)!(n + 1 - k)!} \\ &= \frac{n!}{k!(n - k)!} \cdot \frac{(n + 1)!}{(k + 1)!(n + 1 - k)!} = \frac{\binom{n}{k}\binom{n+1}{k+1}}{n - k + 1}. \end{aligned}$$

Also solved by Herta T. Freitag, Ralph Garfield, Peter A. Lindstrom, C. B. A. Peck, Bob Prielipp, A. G. Shannon, Sahib Singh, Paul Smith, Gregory Wulczyn, and proposer.

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