DEGENERACY OF TRANSFORMED COMPLETE SEQUENCES

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1. INTRODUCTION

A sequence $S = \{s_i\}_{i=1,2,\ldots}$ of natural numbers is said to be *complete* (see [4]) if every positive integer can be represented as the sum of distinct terms of S. If, furthermore, the sequence is nondecreasing and begins with $s_1 = 1$, then a necessary and sufficient condition in order that S be complete is:

$$s_{n+1} \le 1 + \sum_{i=1}^{n} s_i, \qquad \text{for } n > 1 \quad (\text{see [2]}).$$

Note that this condition includes the possibility that some members of S may be equal, thus corresponding to a representation in which certain terms may be repeated, as has been considered [1].

It is shown in [3] that completeness is preserved under certain transformations of S, an example of which is $x \to \langle \ln x \rangle$, where $\ln x$ is the natural logarithm of x and $\langle \ln x \rangle$ is the smallest integer greater than $\ln x$. Two complete sequences to which this transformation is applied are the Fibonacci sequence and the sequence of prime numbers (with 1 included).

We will develop here conditions, on S and the transformation considered, which guarantee that the transformed sequence is degenerate in the sense that it includes all natural numbers (and hence is complete). The above two sequences, discussed in [3], satisfy our conditions.

2. DEGENERACY

To ensure that a transformed sequence be a sequence of natural numbers, as well as complete, we use the following operation:

Definition: For x a real number $\langle x \rangle$ is the smallest integer strictly greater than x.

Lemma: If x, y, and z are real numbers such that $0 \le x - y \le z$, then

$$0 \le \langle x \rangle - \langle y \rangle < z + 1.$$

Proof: By the definition of $\langle \cdot \rangle$,

$$\langle x \rangle - 1 \le x < \langle x \rangle \quad \text{and} \quad \langle y \rangle - 1 \le y < \langle y \rangle,$$

hence, $\quad z \ge x - y > \langle x \rangle - 1 - \langle y \rangle.$

Clearly, $\quad x \ge y \quad$ implies $\quad \langle x \rangle \ge \langle y \rangle.$

We now use this property in the proof of our fundamental result. Note that the sequence S need not be complete.

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Theorem: Let $S = \{s_i\}_{i=1,2,\ldots}$ be a sequence of natural numbers and let f be a real-valued nondecreasing function defined on S such that

$$\lim_{i\to\infty} f(s_i) = \infty.$$

If there exists an I such that for all $i \geq I$, $f(s_{i+1}) - f(s_i) \leq 1$, then the set $A = \{\langle f(s_i)\rangle | i = 1, 2, \ldots\}$ contains $\{M, M+1, \ldots\}$, the set of all integers greater than $M - 1$, where $M = \langle f(s_I)\rangle$.

Proof: For any $i \geq I$, since f is nondecreasing,

$$0 \leq f(s_{i+1}) - f(s_i) \leq 1.$$

Thus, by the lemma

$$0 \leq \langle f(s_{i+1})\rangle - \langle f(s_i)\rangle < 2;$$

that is,

$$\langle f(s_{i+1})\rangle = \begin{cases} \langle f(s_i)\rangle \\ \langle f(s_i)\rangle + 1. \end{cases}$$

But since $\lim_{i\to\infty} f(s_i) = \infty$ the transformed sequence

$$\{\langle f(s_i)\rangle\}_{i=1,2,\ldots}$$

cannot remain eventually constant and so $A \supset \{M, M+1, \ldots\}$.

Of particular interest is the case when $0 \leq f(s_I) < 1$: the set A becomes the set $\mathbb{N}$ of all natural numbers. For example, if

$$S = \{1, 2, 3, 4, 11, 12, 13, \ldots, n, \ldots\}$$

and $f = \ln$, the transformed sequence is $\mathbb{N}$. Note that in this example

$$s_5 = 2.75 s_4 > e \cdot s_4.$$

Corollary 1: Let $S = \{s_i\}_{i=1,2,\ldots}$ be a nondecreasing sequence of integers with $s_1 = 1$ and $\lim_{i\to\infty} s_i = \infty$. If a is a real number > 1 such that $s_{i+1} \leq as$, for all $i \geq 1$, then:

$$\{\langle \ln_b s_i\rangle | i = 1, 2, 3, \ldots\} = \mathbb{N}, \text{ for all } b \geq a.$$

Proof: The conditions of the theorem are met with

$$M = \langle \ln_b 1\rangle = 1$$

and $0 \leq \ln_b s_{i+1} - \ln_b s_i \leq \ln_b a \leq 1$, for all $i \geq 1$.

Example 1: If a is an integer ≥ 2 and

$$s_i = a^{i-1} \text{ for } i = 1, 2, \ldots,$$

then clearly $\{\langle \ln_a a^{i-1}\rangle\} = \mathbb{N}$.

Example 2: If $a = e$ (so $\ln_a$ becomes the natural logarithm ln) and

$$S = \{1, 2, 3, 5, 7, 11, \ldots\},$$

the prime numbers with 1 included, the transformed sequence passes through all natural numbers. The inequality condition of the corollary is satisfied by virtue of Bertrand's postulate $p_{n+1} < 2p_n$, where p_n is the nth prime [5].

Example 3: Again, if $a = e$ but with the sequence S now the sequence of Ficonacci numbers, the image is still $\mathbb{N}$, for

$$F_{n+1} = \frac{(1+\sqrt{5})}{2} \cdot F_n + \left(\frac{1-\sqrt{5}}{2}\right)^n < \left(\frac{1+\sqrt{5}}{2} + 1\right)F_n < eF_n.$$

Example 4: $\{\langle \ln L_n \rangle | n = 1, 2, \ldots\} = \mathbb{N}$ where L_n are the Lucas numbers.

The last three examples in fact satisfy the stronger conditions of:

Corollary 2: Let $S = \{s_i\}_{i=1,2,\ldots}$ be a nondecreasing sequence of integers starting $s_1 \leq 2$, $s_2 \leq 7$, with $\lim_{i\to\infty} s_i = \infty$, and satisfying the inequality

$$s_{n+2} \leq s_{n+1} + s_n, \; n \geq 1.$$

Then $$\{\langle \ln s_i \rangle | i = 1, 2, \ldots\} = \mathbb{N}.$$

Proof: If $s_{n+2} \leq s_{n+1} + s_n$, then $s_{n+2} \leq 2 \cdot s_{n+1} < e \cdot s_{n+1}$.

The theorem guarantees

$$\{\langle \ln s_n \rangle | n = 2, \ldots\} = \{M, M+1, \ldots\}$$

where $$M = \langle \ln s_2 \rangle \leq \langle \ln 7 \rangle = 2.$$

And the condition on s_1 ensures $\langle \ln s_1 \rangle = 1$.

Note that Example 2 satisfies the conditions of Corollary 2 since, for the primes $p_{i+2} \leq p_{i+1} + p_i$. (See, for example [5, p. 139].)

Corollary 3: Let $S = \{s_i\}$, $i = 1, 2, 3, \ldots$ be a nondecreasing sequence of integers with $s_1 = 1$ and $\lim_{i\to\infty} s_i = \infty$. Let a and b be two positive integers such that $s_{i+2} \leq as_{i+1} + bs_i$ for all $i \geq 1$. Then

$$\{\langle \ln_c s_i \rangle | i = 1, 2, 3, \ldots\} = \mathbb{N} \text{ for all } c \geq a + b.$$

Proof: Since $s_{i+2} \leq (a+b)s_{i+1}$ for all $i \geq 1$, we have

$$0 \leq \ln_c s_{i+2} - \ln_c s_{i+1} \leq \ln_c(a+b) \leq 1.$$

Furthermore, $M = \langle \ln_c 1 \rangle = 1$. Hence, all the conditions of the theorem are met.

3. CONCLUDING REMARKS

As can be found in [3], there exist transformations which do not degenerate complete sequences, e.g., the Lucas transformation and the function

$$f(x) = \alpha x, \text{ where } 0 < \alpha < 1.$$

We note that even the quantized logarithmic transformation, $x \to \langle \ln x \rangle$, does not itself produce degeneracy as is shown by a complete sequence that begins 1, 2, 3, 4, 5, 6, 22, ...

Another example of an explicit function which sometimes degenerates sequences is $\Pi(x)$, the number of primes not exceeding the real number x. It is clear that this function degenerates the sequence of primes itself; as would the counting function of S, an arbitrary (countable) sequence, degenerate S. However, the image of the Fibonacci numbers under Π is not $\mathbb{N}$, since $\Pi(8) = 4$ but $\Pi(13) = 6$.

All the sequences above are complete (although repetitions must be permitted in Example 1 if $a > 2$), but the theorem does not assume completeness. We conclude with an example of a sequence which is not complete but by an immediate application of Corollary 1 is seen to be transformed into $\mathbb{N}$ under $f(x) = \ln x$. The sequence in question is $s_1 = 1$, $s_2 = 2$, and for $n \geq 3$, $s_n = 5 \cdot 2^{n-3}$. To see that this sequence is not complete, observe that

$$5 \cdot 2^{n-3} - 1, \text{ for } n \geq 3,$$

can never be expressed as the sum of distinct terms of the sequence.

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FINDING THE GENERAL SOLUTION OF A LINEAR DIOPHANTINE EQUATION

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ABSTRACT

A new procedure for finding the general solution of a linear diophantine equation is given. As a byproduct, the algorithm finds the greatest common divisor (gcd) of a set of integers. Related results and discussion concerning existing procedures are also given.

1. INTRODUCTION

This note presents an alternative procedure for computing the greatest common divisor of a set of n integers $a_1, a_2, \ldots, a_n$, denoted by

$$\gcd(a_1, a_2, \ldots, a_n),$$

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