as it can be readily shown that

$$U_{1,k+2}^{(2)} = -P_{22}U_{2,k+1}^{(2)}.$$

The first five values of $U_{j,n}^{(2)}$, $j = 1, 2$, are:

$U_{j,n}^{(2)}$	$n = 1$	2	3	4	5
$j = 1$	1	0	$-P_{22}$	$-P_{21}P_{22}$	$-P_{21}^2P_{22} + P_{22}^2$
2	0	1	P_{21}	$P_{21}^2 - P_{22}$	$P_{21}^3 - 2P_{21}P_{22}$

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GENERALIZED FIBONACCI NUMBERS AS ELEMENTS OF IDEALS

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Wyler [3] has looked at the structure of second-order recurrences by considering them as elements of a commutative ring with the Lucas recurrence as unit element.

It is possible to supplement Wyler's results and to gain further insight into the structure of recurrences by looking at ideals in this commutative ring.

The purpose of this note is to look briefly at the structure of Horadam's generalized sequence of numbers [2] defined recursively by

$$w_n = pw_{n-1} - qw_{n-2} \quad (n \geq 2) \tag{1}$$

with $w_0 = a$, $w_1 = b$, and where p,q are arbitrary integers.

DeCarli [1] has examined a similarly generalized sequence over an arbitrary ring. It is proposed here to assume that the sequence $\{w_n\}$ of numbers are elements of a commutative ring R and to examine $\{w_n\}$ in terms of ideals of R. To this end, suppose that p,q are elements of an ideal of R.

$\langle p \rangle, \langle q \rangle$ are then the ideals generated by p and q, respectively, and $(\langle p \rangle, \langle q \rangle)$ is the sum of the ideals generated by p and q.

Theorem 1: $w_n \in (\langle p\rangle, \langle q\rangle)$.

Proof: $a, b \in R$, $p \in \langle p\rangle$, $q \in \langle q\rangle$ implies $pb \in \langle p\rangle$, and $-qa \in \langle q\rangle$.

$$\therefore\ w_2 = pb - qa \in \langle p\rangle + \langle q\rangle.$$

$pb \in \langle p\rangle$ and so $pb \in R$.

Hence $-q(pb) \in \langle q\rangle$.

$w_2 \in R$, $p \in \langle p\rangle$ implies $pw_2 \in \langle p\rangle$.

$w_3 = pw_2 - qb \in \langle p\rangle + \langle q\rangle$.

It follows by induction that $w_n \in \langle p\rangle + \langle q\rangle$: that is,

$$w_n \in (\langle p\rangle, \langle q\rangle).$$

The general term of $\{w_n\}$ can be expressed in terms of

$$\alpha = \frac{1}{2}\left(p + \sqrt{(p^2 - 4q)}\right) \quad \text{and} \quad \beta = \frac{1}{2}\left(p - \sqrt{(p^2 - 4q)}\right)$$

as follows:

(2) $$w_n = A\alpha^n + B\beta^n$$

where $A = (b - a\beta)/(\alpha - \beta)$ and $B = (a\alpha - b)/(\alpha - \beta)$.

Suppose A, B are elements of a commutative ring Q, and α, β are elements of an ideal of Q. It follows that w_n belongs to Q and is also a member of the sum of the ideals generated by α and β (from Theorem 1).

Theorem 2: $\exists S \subset Q$ such that $S = \langle\alpha\rangle \oplus \langle\beta\rangle$ if $p^2 - 4q \neq 0$.

Proof: $\alpha^i = \beta^j$ iff $p^2 - 4q = 0$. Hence,

$$\langle\alpha\rangle \cap \langle\beta\rangle = \langle 0\rangle,$$

and the result follows.

Let $\{v_n\}$ and $\{w_n\}$ be two sequences of elements of Q such that

(3) $$v_n = p'v_{n-1} - q'v_{n-2}$$

(with suitable initial values and p', q' arbitrary integers) and w_n is defined as before.

Define $\{v_m\} \equiv \{w_n\}$ when $w_n - v_m \in S$ for small n, m where

$$S = \langle\alpha\rangle \oplus \langle\beta\rangle$$

as before.

Note that if $a, b, c \in S$, then (i) $a - a \in S$; (ii) $a - b \in S$ implies $b - a \in S$; (iii) $a - b \in S$ and $b - c \in S$ imply that $a - c \in S$.

Theorem 3: If $v_m - w_n \in \langle\alpha\rangle \oplus \langle\beta\rangle$ for small n, m, then

$$v_m - w_n \in \langle\alpha\rangle \oplus \langle\beta\rangle \text{ for all } n, m.$$

Proof: $w_n \in S = \alpha \oplus \beta$ for all n from Theorem 2. It is known that $v_m \in S$ for $m \leq N$, say. Now,

$$p'v_N \in S \quad \text{and} \quad q'v_{N-1} \in S.$$

Hence, $v_{N+1} = p'v_N - q'v_{N-1} \in S$, and the result follows.

To prove the stronger result that if

$$\{v_m\} \equiv \{w_n\} \text{ for any } n,m, \text{ then } \{v_m\} \equiv \{w_n\} \text{ for all } n,m,$$

it would be necessary to replace "small" with "large" in the enunciation of Theorem 3. This would require S to be a prime ideal which could be achieved by embedding S in a maximal ideal $\mathcal{M}\alpha\beta$ which could be proved prime. However, this would then require restrictions on p' and q' as it would be easy to show that $q'v_{N-1} \in S$ but it would not automatically follow that $v_{N-1} \in S$.

Another problem that might be worth investigating is to look for commutators for relations like

$$w_{n+1}^p - w_n^p - w_{n-1}^p, \text{ where } p \text{ is a prime.}$$

These could be useful in Lie algebras.

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A GENERALIZATION OF HILTON'S PARTITION OF HORADAM'S SEQUENCES

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1. INTRODUCTION

If $P_{r1}, P_{r2}, \ldots, P_{rr}$ are distinct integers for positive r, let

$$\omega = \omega(P_{r1}, \ldots, P_{rr})$$

be the set of integer sequences

$$\left\{W_{sn}^{(r)}\right\} = \left\{W_{s0}^{(r)}, W_{s1}^{(r)}, W_{s2}^{(r)}, \ldots\right\}$$

which satisfy the recurrence relation of order r,

$$W_{s,n+r}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} W_{s,n+r-j}^{(r)}, \quad (s = 1, 2, \ldots, r), \quad n \geq 1. \tag{1.1}$$

This is a generalization of of $\left\{W_{sn}^{(2)}\right\}$ studied in detail by Horadam [1, 2, 3, 4, 5].

Hilton [6] partitioned Horadam's sequence into a set F of generalized Fibonacci sequences and a set L of generalized Lucas sequences. We extend this to show that ω can be partitioned naturally into r sets of generalized sequences.