6637

RECURRENCES FOR TWO RESTRICTED PARTITION FUNCTIONS

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In this note we shall develop two "pure" recurrences for determination of the functional values $q(n)$ and $q_0(n)$. Accordingly, we recall that for a given natural number n, $q(n)$ denotes the number of partitions of n into distinct parts (or, equivalently, the number of partitions of n into odd parts), and $q_0(n)$ denotes the number of partitions of n into distinct odd parts (or, equivalently, the number of self-conjugate partitions of n). As usual, $p(n)$ denotes the number of unrestricted partitions of n; and, conventionally, we set $p(0) = q(0) = q_0(0) = 1$. Previous tables of values for $q_0(n)$ and $q(n)$ have been constructed on the strength of known tables for $p(n)$; for example, see [1] and [3]. The recurrences of the following two theorems allow us to determine $q_0(n)$ and $q(n)$ without prior knowledge of $p(n)$.

Theorem 1: For each nonnegative integer n,

$$\sum_{k=0} (-1)^{k(k+1)/2} \cdot q_0(n - k(k+1)/2) = \begin{cases} (-1)^m, & \text{if } n = m(3m \pm 1) \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Theorem 2: For each nonnegative integer n,

$$q(n) + 2\sum_{k=1} (-1)^k \cdot q(n - k^2) = \begin{cases} (-1)^m, & \text{if } n = m(3m \pm 1)/2 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

In both theorems, summation is extended over all values of the indices which yield nonnegative integral arguments of q_0 and q.

Our proofs will depend on the following three identities of Euler and Gauss [2, p. 284]:

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{n=1}^{\infty} (-1)^n \left\{x^{(3n^2-n)/2} + x^{(3n^2+n)/2}\right\}. \tag{3}$$

$$\prod_{n=1}^{\infty} (1 - x^{2n}) = \prod_{n=1}^{\infty} (1 + x^{2n-1}) \cdot \sum_{n=0}^{\infty} (-x)^{n(n+1)/2}. \tag{4}$$

$$\prod_{n=1}^{\infty} (1 - x^n) = \prod_{n=1}^{\infty} (1 + x^n) \left\{1 + 2\sum_{n=1}^{\infty} (-1)^n \cdot x^{n^2}\right\}. \tag{5}$$

Proof of Theorem 1: Replace x by x^2 in (3) and eliminate $\Pi(1 - x^{2n})$ between the resulting identity and (4) to obtain

$$\sum_{n=0}^{\infty} q_0(n)x^n \cdot \sum_{n=0}^{\infty} (-x)^{n(n+1)/2} = 1 + \sum_{m=1}^{\infty} (-1)^m \left\{x^{3m^2-m} + x^{3m^2+m}\right\}.$$

[Recall that $\Pi(1 + x^{2n-1})$ generates $q_0(n)$.] The complete expansion of the left side of the foregoing equation is:

$$\sum_{n=0}^{\infty} x^n \sum_{k=0} (-1)^{k(k+1)/2} q_0(n - k(k+1)/2).$$

Equating coefficients of x^n, we obtain the desired conclusion. [Note that $q_0(0) = 1$ is consistent with the statement of our theorem.]

Proof of Theorem 2: In view of the fact that $\Pi(1 + x^n)$ generates $q(n)$, identities (3) and (5) imply

$$\left\{\sum_{n=0}^{\infty} q(n)x^n\right\}\left\{1 + 2\sum_{n=1}^{\infty}(-1)^n x^{n^2}\right\} = 1 + \sum_{m=1}^{\infty}(-1)^m\left\{x^{(3m^2-m)/2} + x^{(3m^2+m)/2}\right\},$$

or, equivalently,

$$\sum_{n=0}^{\infty} x^n\left\{q(n) + \sum_{k=1}(-1)^k \cdot 2q(n - k^2)\right\} = 1 + \sum_{m=1}^{\infty}(-1)^m\left\{x^{(3m^2-m)/2} + x^{(3m^2+m)/2}\right\}.$$

Upon equating coefficients of x^n, we derive the recurrence.

REMARKS

The following table of values for $q_0(n)$, $q(n)$, and $p(n)$, $n = 0(1)25$, is included to show the relative rates of growth of the three functions. For example, $q_0(n)$ grows much more slowly with n than does $p(n)$. So, computing a list of values of $q_0(n)$ by using "large" $p(n)$ values is much less desirable than by use of the recurrence (1).

TABLE 1

n	$q_0(n)$	$q(n)$	$p(n)$	n	$q_0(n)$	$q(n)$	$p(n)$
0	1	1	1	13	3	18	101
1	1	1	1	14	3	22	135
2	0	1	2	15	4	27	176
3	1	2	3	16	5	32	231
4	1	2	5	17	5	38	297
5	1	3	7	18	5	46	385
6	1	4	11	19	6	54	490
7	1	5	15	20	7	64	627
8	2	6	22	21	8	76	792
9	2	7	30	22	8	89	1002
10	2	10	42	23	9	104	1255
11	2	12	56	24	11	122	1575
12	3	15	77	25	12	142	1958

REFERENCES

1. J. A. Ewell. "Partition Recurrences." *J. Combinatorial Theory*, Ser. A, 14 (1973):125-127.
2. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: Clarendon Press, 1960.
3. G. N. Watson. "Two Tables of Partitions." *Proc. London Math. Soc.* (2), 42 (1937):550-556.

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