EXTENSIONS OF A PAPER ON DIAGONAL FUNCTIONS

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INTRODUCTION

Consider the sequences $\{A_n(x)\}$ and $\{B_n(x)\}$ for which

(1) $$A_{n+2}(x) = pxA_{n+1}(x) + qA_n(x), \quad A_0(x) = 0,\ A_1(x) = 1;$$

(2) $$B_{n+2}(x) = pxB_{n+1}(x) + qB_n(x), \quad B_0(x) = 2,\ B_1(x) = x.$$

Then, from (1) and (2), we have

(3) $$\left\{\begin{array}{l} A_0(x) = 0 \\ A_1(x) = 1 \\ A_2(x) = px \\ A_3(x) = p^2x^2 + q \\ A_4(x) = p^3x^3 + 2pqx \\ A_5(x) = p^4x^4 + 3p^2qx^2 + q^2 \\ A_6(x) = p^5x^5 + 4p^3qx^3 + 3pq^2x \\ A_7(x) = p^6x^6 + 5p^4qx^4 + 6p^2q^2x^2 + q^3 \\ A_8(x) = p^7x^7 + 6p^5qx^5 + 10p^3q^2x^3 + 4pq^3x \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right.$$

(4) $$\left\{\begin{array}{l} B_0(x) = 2 \\ B_1(x) = px \\ B_2(x) = p^2x^2 + 2q \\ B_3(x) = p^3x^3 + 3pqx \\ B_4(x) = p^4x^4 + 4p^2qx^2 + 2q^2 \\ B_5(x) = p^5x^5 + 5p^3qx^3 + 5pq^2x \\ B_6(x) = p^6x^6 + 6p^4qx^4 + 9p^2q^2x^2 + 2q^3 \\ B_7(x) = p^7x^7 + 7p^5qx^5 + 14p^3q^2x^3 + 7pq^3x \\ B_8(x) = p^8x^8 + 8p^6qx^6 + 20p^4q^2x^4 + 16p^2q^3x^2 + 2q^4 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right.$$

In this paper we seek to extend and generalize the results of [1], [2], [3], [4], and Jaiswal [5]. The results hereunder flow on from those in [2], where certain restrictions were imposed on the sequences for the purpose of extending the results of Serkland [6].

DIAGONAL FUNCTIONS FOR $A_n(x)$, $B_n(x)$

Label the rising and descending diagonal functions of x $R_i(x)$ and $D_i(x)$ for $\{A_n(x)\}$, and $r_i(x)$ and $d_i(x)$ for $\{B_n(x)\}$.

From (3) and (4), we readily obtain

$$(5)\qquad \begin{cases} R_1(x) = 1 \\ R_2(x) = px \\ R_3(x) = p^2x^2 \\ R_4(x) = p^3x^3 + q \\ R_5(x) = p^4x^4 + 2pqx \\ R_6(x) = p^5x^5 + 3p^2qx^2 \\ R_7(x) = p^6x^6 + 4p^3qx^3 + q^2 \\ R_8(x) = p^7x^7 + 5p^4qx^4 + 3pq^2x \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

$$(6)\qquad \begin{cases} r_1(x) = 2 \\ r_2(x) = px \\ r_3(x) = p^2x^2 \\ r_4(x) = p^3x^3 + 2q \\ r_5(x) = p^4x^4 + 3pqx \\ r_6(x) = p^5x^5 + 4p^2qx^2 \\ r_7(x) = p^6x^6 + 5p^3qx^3 + 2q^2 \\ r_8(x) = p^7x^7 + 6p^4qx^4 + 5pq^2x \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

with the properties ($n > 3$)

$$(7)\qquad \begin{cases} r_n(x) = R_n(x) + qR_{n-3}(x) \\ R_n(x) = pxR_{n-1}(x) + qR_{n-3}(x) \\ r_n(x) = xr_n(x) + qr_{n-3}(x) \end{cases}$$

Further, we have from (3) and (4), after some simplification in (8),

$$(8)\qquad \begin{cases} D_1(x) = 1 \\ D_2(x) = px + q \\ D_3(x) = (px + q)^2 \\ D_4(x) = (px + q)^3 \\ D_5(x) = (px + q)^4 \\ D_6(x) = (px + q)^5 \\ \ldots\ldots\ldots\ldots \end{cases}$$

and

(9) $$\begin{cases} d_1(x) = 2 \\ d_2(x) = px + 2q \\ d_3(x) = (px + 2q)(px + q) \\ d_4(x) = (px + 2q)(px + q)^2 \\ d_5(x) = (px + 2q)(px + q)^3 \\ d_6(x) = (px + 2q)(px + q)^4 \\ \cdots\cdots\cdots\cdots \end{cases}$$

whence

(10) $$D_n(x) = (px + q)^{n-1} \qquad (n \geq 1)$$

(11) $$d_n(x) = (px + 2q)(px + q)^{n-2} \qquad (n \geq 2)$$

(12) $$d_n(x) = D_n(x) + qD_{n-1}(x) \qquad (n \geq 2)$$

giving

(13) $$\frac{D_{n+1}(x)}{D_n(x)} = \frac{d_{n+1}(x)}{d_n(x)} = px + q$$

(14) $$\frac{d_{n+1}(x)}{D_n(x)} = px + 2q$$

(15) $$\frac{d_n(x)}{D_n(x)} = \frac{px + 2q}{px + q} \qquad (px + q \neq 0)$$

GENERATING FUNCTIONS FOR THE DIAGONAL FUNCTIONS

Generating functions for the descending diagonal functions are found to be

(16) $$\sum_{n=1}^{\infty} D_n(x)t^{n-1} = [1 - (px + q)t]^{-1}$$

(17) $$\sum_{n=2}^{\infty} d_n(x)t^{n-2} = (px + 2q)[1 - (px + q)t]^{-1}$$

while those for the rising diagonal functions are

(18) $$\sum_{n=1}^{\infty} R_n(x)t^{n-1} = [1 - (pxt + qt^3)]^{-1}$$

(19) $$\sum_{n=2}^{\infty} r_n(x)t^{n-1} = (1 + qt^3)[1 - (pxt + qt^3)]^{-1}.$$

SOME PROPERTIES INVOLVING DIFFERENTIAL EQUATIONS

Limiting ourselves to the types of results studied by Jaiswal [5], let us write, for convenience,

(20) $$D \equiv D(x,t) = \sum_{n=1}^{\infty} D_n(x)t^{n-1}$$

(21) $$d \equiv d(x,t) = \sum_{n=2}^{\infty} d_n(x)t^{n-2}.$$

Calculations using (16) and (17) and the notation of (20) and (21) then lead to the following differential equations involving the descending diagonal functions:

(22) $$pt\frac{\partial D}{\partial t} - (px + q)\frac{\partial D}{\partial x} = 0$$

(23) $$pt\frac{\partial d}{\partial t} - (px + q)\left[\frac{\partial d}{\partial x} - pD\right] = 0$$

(24) $$(px + q)\frac{d}{dx}D_n(x) = p(n - 1)D_n(x)$$

(25) $$(px + q)\frac{d}{dx}[d_{n+2}(x)] - p(n + 1)d_{n+2}(x) + pq(px + q)D_n(x) = 0.$$

Write

(26) $$R \equiv R(x,t) = \sum_{n=1}^{\infty} R_n(x)t^{n-1}$$

(27) $$r \equiv r(x,t) = \sum_{n=2}^{\infty} r_n(x)t^{n-1}.$$

Corresponding differential equations for the rising diagonal functions are, by (18), (19), (26), and (27):

(28) $$pt\frac{\partial R}{\partial t} - (px + 3qt^2)\frac{\partial R}{\partial x} = 0$$

(29) $$pt\frac{\partial r}{\partial t} - (px + 3qt^2)\frac{\partial r}{\partial x} - 3p(r - R) = 0$$

(30) $$px\frac{d}{dx}R_{n+2}(x) + 3q\frac{d}{dx}R_n(x) - p(n + 1)R_{n+2}(x) = 0$$

(31) $$px\frac{d}{dx}r_{n+2}(x) + 3q\frac{d}{dx}r_n(x) - p(n - 2)r_{n+2}(x) - 3pR_{n+2}(x) = 0.$$

Explicit formulation of expressions for $R_{n+1}(x)$ and $r_{n+1}(x)$ can be obtained by comparison of coefficients of t^n in (18) and (19), respectively.

Computation gives

(32) $$R_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i}(px)^{n-3i}q^i$$

(33) $$r_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i}(px)^{n-3i}q^i + \sum_{i=0}^{[(n-3)/3]} \binom{n-3-2i}{i}(px)^{n-3-3i}q^{i+1},$$

where $[n/3]$ means the integral part of $n/3$.

SOME SPECIAL CASES

Contents of the several papers mentioned in the introduction have thus been generalized, *mutatis mutandis*.

If $p = 1$, $q = 1$, the results of [2] are obtained, including the special cases of the *Fibonacci*, *Lucas*, and *Pell* sequences.

If $p = 2$, $q = -1$, the results of [1] and [4], and of Jaiswal [5], follow for the *Chebyshev* polynomial sequences.

Observe that, for the Chebyshev polynomials of the first kind $U_n(x)$, it is customary (e.g., in [1], [4], and [5]) to define $U_0(x) = 1$, $U_1(x) = 2x$; whereas, from (1), the corresponding generalized forms require

$$A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = px,\ \ldots .$$

For our purposes, this is unimportant. However, suitable adjustments can be made if desired.

If $p = 1$, $q = -2$, the results of [4] for the *Fermat* polynomial sequences follow.

THE FERMAT SEQUENCES

For the record, the following results, which were left to the reader's curiosity in [4], are listed (using the symbolism of [4]).

Differential equation properties

$$t\frac{\partial D}{\partial t} - (x - 2)\frac{\partial D}{\partial x} = 0 \tag{34}$$

$$t\frac{\partial D'}{\partial t} - (x - 2)\left\{\frac{\partial D'}{\partial x} - D\right\} = 0 \tag{35}$$

$$(x - 2)\frac{dD(x)}{dx} = (n - 1)D_n(x) \tag{36}$$

$$(x - 2)\frac{d}{dx}[D'_{n+2}(x)] - (n + 1)D'_{n+2}(x) - 2(x - 2)D_n(x) = 0 \tag{37}$$

with corresponding equations for the rising diagonal functions

$$t\frac{\partial R}{\partial t} - (x - 6t^2)\frac{\partial R}{\partial x} = 0 \tag{38}$$

$$t\frac{\partial R'}{\partial t} - (x - 6t^2)\frac{\partial R'}{\partial x} - 3(R' - R) = 0 \tag{39}$$

$$x\frac{dR_{n+2}(x)}{dx} - 6\frac{dR_n(x)}{dx} - (n + 1)R_{n+2}(n) = 0 \tag{40}$$

$$x\frac{dR'_{n+2}(x)}{dx} - 6\frac{dR'_n(x)}{dx} - (n - 2)R'_{n+2}(x) - 3R_{n+2}(x) = 0 \tag{41}$$

where the primes in $D'_{n+2}(x)$, $R'_{n+2}(x)$, etc., do not indicate derivatives, and where

$$D' \equiv D'(x,t) = \sum_{n=2}^{\infty} D_n'(x) t^{n-2}$$

and

$$R' \equiv R'(x,t) = \sum_{n=2}^{\infty} R_n'(x) t^{n-1}.$$

Explicit formulation

Employing the method used to obtain (32) and (33), we calculate

$$R_{n+1}(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i} x^{n-3i} (-2)^i \tag{42}$$

$$R_{n+1}'(x) = \sum_{i=0}^{[n/3]} \binom{n-2i}{i} x^{n-3i} (-2)^i + \sum_{i=0}^{[(n-3)/3]} \binom{n-3-2i}{i} x^{n-3-3i} (-2)^{i+1}. \tag{43}$$

CONCLUDING REMARKS

Undoubtedly, there are many other facets of this work remaining to be explored. Suffice it for us to comment here that some basic features of many interesting polynomial sequences have been unified.

Finally, it might be noted that our classification here, in (1) and (2) of the sequence, say $\{W_n(x)\}$, for which $W_{n+2}(x) = pxW_{n+1}(x) + qW_n$, into its Fibonacci-type and Lucas-type components (see [2] for the case $p = 1$, $q = 1$) recalls the article by A. J. W. Hilton entitled "On the Partition of Horadam's Generalized Sequences into Generalized Fibonacci and Lucas Sequences" which appeared in this journal, Vol. 12, No. 4 (1974):239-245.

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