A NOTE ON TILING RECTANGLES WITH DOMINOES

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INTRODUCTION

In how many ways can an $m \times n$ chessboard be covered by dominoes, each of which covers two adjacent squares? For general m and n this is the "dimer problem" which is known to be difficult (see [2] for details). However, when one of the dimensions, say m, is small, some results can be obtained, and will be given in this paper. The method used has some similarities with that used for the cell-growth problem in [3], although there are differences.

1. THE METHOD

We shall illustrate the general procedure by referring to the case $m = 3$. Any covering of a $3 \times n$ rectangle with dominoes can be regarded as having been built up, domino by domino, in a standard way, starting at the left-hand edge of the rectangle. Each domino is placed so that it covers an uncovered square furthest to the left, and, if there is more than one such square, it covers the one nearest the "top" of the board. Thus if the construction of a covering has proceeded as far as the stage shown in Figure 1, the next domino must be placed so as to cover the position marked with an asterisk. There may be two ways of placing the new domino (as in Figure 1), but there will be only one way if the space below the asterisk is already covered.

In the course of constructing $3 \times n$ rectangles, the figures produced will have irregular right-hand ends—their "profiles." We start by listing the possible profiles and the ways in which one profile can be converted to another by adding an extra domino. This information is given in Figure 2, in which the profiles have been labelled A to I.

Let A_r, B_r, etc., denote the numbers of ways of obtaining figures ending in profiles A, B, etc., by assembling r dominoes. Then, by reference to Figure 2, we obtain the equations:

$$(1.1) \qquad \begin{cases} A_{r+1} = D_r + E_r + F_r \\ B_{r+1} = A_r \\ C_{r+1} = A_r + H_r \\ D_{r+1} = B_r \\ E_{r+1} = B_r + I_r \\ F_{r+1} = C_r \\ G_{r+1} = E_r \\ H_{r+1} = F_r \\ I_{r+1} = G_r \end{cases}$$

Since $A_0 = 1$ and all other values are 0 when $r = 0$, we can use (1.1) to calculate these numbers, and in particular A_r, for $r = 1, 2,$ etc. Equations (1.1) can also be transformed in an obvious way to an equation which expresses the vector $(A_{r+1}, B_{r+1}, \ldots, I_{r+1})$ as a 9×9 matrix times the vector $(A_r, B_r, \ldots, I_r)$, but this is not very useful.

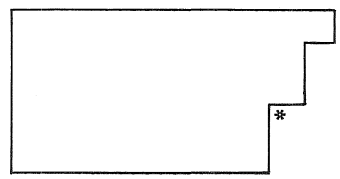

FIGURE 1

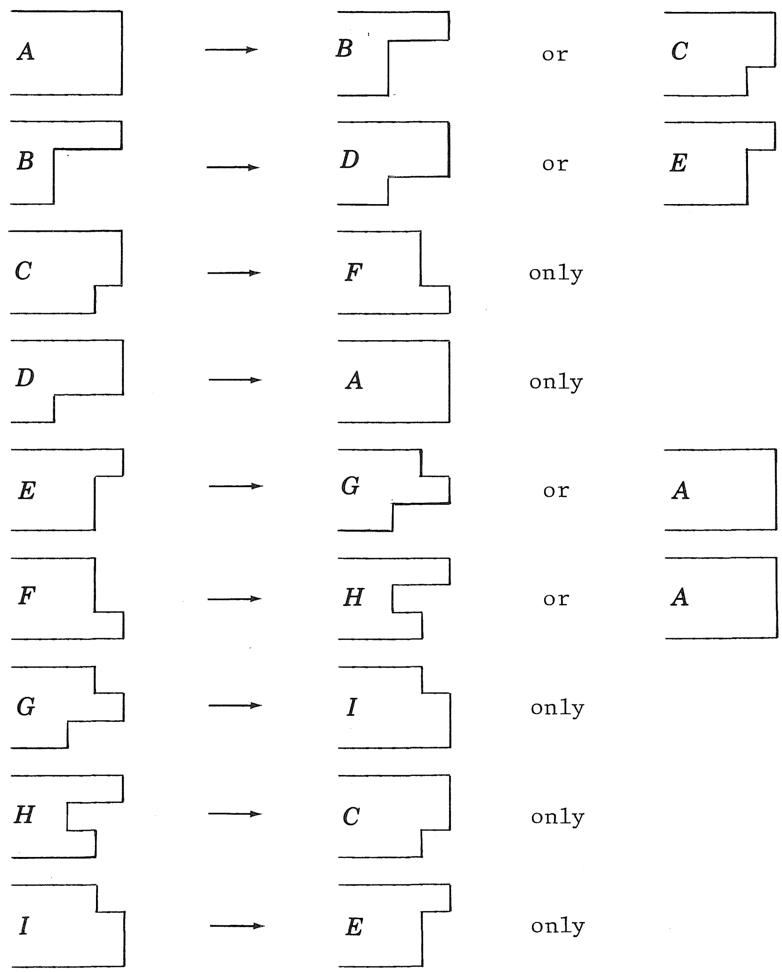

FIGURE 2

A better approach is to define generating functions

$$A(t) = \sum_{r=0}^{\infty} A_r t^r, \text{ etc.}$$

Remembering that $A(t)$ will be the only one of these functions having a constant term, we obtain the relations

$$(1.2)\qquad \begin{cases} A(t) = 1 + tD(t) + tE(t) + tF(t) \\ B(t) = tA(t) \\ C(t) = tA(t) + tH(t) \\ D(t) = tB(t) \\ E(t) = tB(t) + tI(t) \\ F(t) = tC(t) \\ G(t) = tE(t) \\ H(t) = tF(t) \\ I(t) = tG(t) \end{cases}$$

Solving these equations for $A(t)$ we obtain

$$(1 - 4t^3 + t^6)A(t) = 1 - t^3,$$

which can be more conveniently expressed as

$$(1.3)\qquad (1 - 4x + x^2)A(x) = 1 - x,$$

writing $A(x) = \sum_{r=0}^{\infty} a_r x^r$ where $a_r = A_{3r}$. (Clearly $A_k = 0$ if k is not a multiple of 3.)

From (1.3), we find that

$$a_r = 4a_{r-1} - a_{r-2}.$$

2. RESULTS

When $m = 2$, there are two profiles (A and B of Figure 2, with the bottom row omitted) and the corresponding equations are

$$A(t) = 1 + tA(t) + tB(t)$$

$$B(t) = tA(t)$$

whence $A(t) = (1 - t - t^2)^{-1}$. The numbers of tilings are therefore the Fibonacci numbers.

When $m = 4$, the profiles are as shown in Figure 3 and by following the method of Section 1, we obtain the equations

$$\begin{aligned} A(t) &= 1 + tC(t) + tG(t) + tH(t) + tI(t) \\ B(t) &= tA(t);\; C(t) = tA(t) + tD(t) + tK(t) \\ D(t) &= tB(t);\; E(t) = tB(t) + tL(t) \\ F(t) &= tC(t);\; G(t) = tD(t);\; H(t) = tE(t) \\ I(t) &= tF(t);\; J(t) = tH(t);\; K(t) = tI(t);\; L(t) = tJ(t) \end{aligned}$$

from which, on solving for $A(t)$, we obtain

$$A(x) = (1 - x^2)/(1 - x - 5x^2 - x^3 + x^4)$$

and the corresponding recursive formula

$$a_{r+1} = a_r + 5a_{r-1} + a_{r-2} - a_{r-3}.$$

For $m > 4$, the method becomes tedious by hand, but I found it quite easy to write a program (in APL) which would first generate the relations between the profiles (as in Figure 2) and then calculate the required numbers from the equations analogous to (1.1). In this way, results were obtained for m = 5, 6, 7, 8, and 9. They are given in Table 1 below. Note that Kasteleyn [1] has given results for $m = n$ = 2, 4, 6, and 8, with which the entries in the table agree.

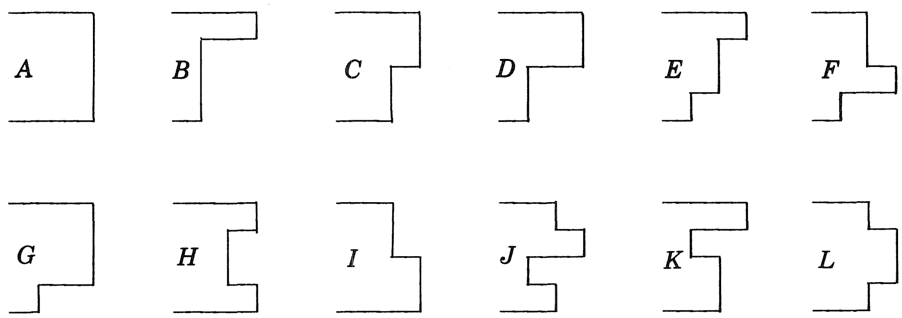

FIGURE 3

TABLE 1

$n \backslash m$	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0
2	2	3	5	8	13	21	34	55
3	3	0	11	0	41	0	153	0
4	5	11	36	95	281	781	2245	6336
5	8	0	95	0	1183	0	14824	0
6	13	41	281	1183	6728	31529	167089	817991
7	21	0	781	0	31529	0	1292697	0
8	34	153	2245	14824	167089	1292697	12988816	108435745
9	55	0	6336	0	817991	0	108435745	0
10	89	571	10861	185921	4213133	53175517	1031151241	14479521761
11	144	0	51205	0	21001799	0	8940739824	0
12	233	2131	145601	2332097	106912793	2188978117	82741005829	1937528668711
13	377	0	413351	0	536948224	0	731164253833	0
14	610	7953	1174500	29253160	2720246633	90124167441	6675498237130	259423766712000
15	987	0	3335651	0	13704300553	0	59554200469113	
16	1597	29681	9475901	366944287	69289288909	3710708201969	540061286536921	
17	2584	0	26915305	0	349519610713	0	4841110033666048	

$n \backslash m$	2	3	4	5	6	7
18	4181	110771	76455961	4602858719	1765711581057	152783289861989
19	6765	0	217172736	0	8911652846951	0
20	10946	413403	616891945	57737128904	45005025662792	6290652543875133
21	17711	0	1752296281	0	227191499132401	0
22	28657	1542841	4977472781	724240365697	1147185247901449	
23	46368	0	14138673395	0	5791672851807479	
24	75025	5757961	40161441636	9084693297025		
25	121393	0	114079985111	0		
26	196418	21489003	324048393905	113956161827912		
27	317811	0	920471087701	0		
28	514229	80198051	2614631600701	1429438110270431		
29	832040	0	7426955448000	0		
30	1346269	299303201	21096536145301			

REFERENCES

1. P. W. Kasteleyn. "Graph Theory and Crystal Physics." In *Graph Theory and Theoretical Physics*, ed. by F. Harary, Ch. 2. New York: Academic Press, 1967.
2. J. K. Percus. *Combinatorial Methods*. Applied Mathematical Sciences, Col. 4. New York: Springer, 1971.
3. R. C. Read. "Contributions to the Cell-Growth Problem." *Canad. J. Math.* 14 (1962):1-20.

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