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SOME LACUNARY RECURRENCE RELATIONS

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1. INTRODUCTION

Kirkpatrick [4] has discussed aspects of linear recurrence relations which skip terms in a Fibonacci context. Such recurrence relations are called "lacunary" because there are gaps in them where they skip terms. In the same issue of this journal, Berzsenyi [1] posed a problem, a solution of which is also a lacunary recurrence relation. These are two instances of a not infrequent occurrence.

We consider here some lacunary recurrence relations associated with sequences $\{w_n^{(r)}\}$, the elements of which satisfy the linear homogeneous recurrence relation of order r:

$$w_n^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, w_{n-j}^{(r)}, \quad n > r,$$

with suitable initial conditions, where the P_{rj} are arbitrary integers. The sequence, $\{v_n^{(r)}\}$, with initial conditions given by

$$v_n^{(r)} = \begin{cases} 0 & n < 0, \\ \sum_{j=1}^{r} \alpha_{rj}^n & 0 \le n < r \end{cases}$$

is called the "primordial" sequence, because when $r = 2$, it becomes the primordial sequence of Lucas [6]. The α_{rj} are the roots, assumed distinct, of the auxiliary equation

$$x^r = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, x^{r-j}.$$

We need an arithmetical function $\delta(m,s)$ defined by

$$\delta(m,s) = \begin{cases} 1 & \text{if } m \mid s, \\ 0 & \text{if } m \nmid s. \end{cases}$$

We also need $s(r,m,j)$, the symmetric functions of the α_{ri}^m, $i = 1, 2, \ldots, r$, taken j at a time, as in Macmahon [5]:

$$s(r,m,j) = \Sigma \alpha_{ri_1}^m \alpha_{ri_2}^m \ldots \alpha_{ri_j}^m,$$

in which the sum is over a distinct cycle of α_{ri}^m taken j at a time and where we set $s(r,m,0) = 1$.

For example,

$$s(3,m,1) = \alpha_{31}^m + \alpha_{32}^m + \alpha_{33}^m,$$
$$s(3,m,2) = (\alpha_{31}\alpha_{32})^m + (\alpha_{32}\alpha_{33})^m + (\alpha_{33}\alpha_{31})^m,$$
$$s(3,m,3) = (\alpha_{31}\alpha_{32}\alpha_{33})^m;$$
$$s(r,m,1) = v_m^{(r)},$$
$$s(r,1,j) = P_{rj}$$
$$s(r,m,r) = P_{rr}^m.$$

2. PRIMORDIAL SEQUENCE

Lemma 1: For $m \geq 0$,

$$\sum_{n=0}^{\infty} v_{(n+1)m}^{(r)} x^n = \left(\sum_{j=1}^{n+1} js(r,m,j)(-x)^{j-1}\right) \Bigg/ \left(\sum_{j=0}^{r} (-1)^j s(r,m,j) x^j\right).$$

Proof:
$$\sum_{n=0}^{\infty} v_{(n+1)m}^{(r)} x^n = \sum_{n=0}^{\infty} \sum_{i=1}^{r} \alpha_{ri}^{nm+m} x^n$$
$$= \sum_{i=1}^{r} \alpha_{ri}^m \sum_{n=0}^{\infty} (\alpha_{ri}^{mn} x)^n = \sum_{i=1}^{r} \alpha_{ri}^m (1 - \alpha_{ri}^m x)^{-1}$$
$$= \sum_{i=1}^{r} \alpha_{ri}^m \prod_{\substack{j=1 \\ j \neq 1}}^{r} (1 - \alpha_{rj}^m x) \Big/ \prod_{j=1}^{r} (1 - \alpha_{rj}^m x)$$
$$= \frac{\displaystyle\sum_{i=1}^{r} \alpha_{ri} - \sum_{\substack{j=1 \\ j \neq 1}}^{r} \alpha_{ri}^m \alpha_{rj}^m x + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{r} \alpha_{ri}^m \alpha_{rj}^m \alpha_{rk}^m x^2 - \cdots}{\displaystyle\prod^{r} (1 - \alpha_{rj}^m x)}$$

$$= \frac{s(r,m,1) - 2s(r,m,2)x + 3s(r,m,3)x^2 - \cdots}{\displaystyle\sum_{j=0}^{r} (-1)^j s(r,m,j) x^j}$$

because each α_{ri}, $i = 1, 2, \ldots, j \leq r$ moves through j positions in a complete cycle.

Examples of the lemma when $r = 2$ are obtained by comparing the coefficients of x^n in

$$\sum_{n=0}^{\infty} (-1)^n s(r,m,n) x^n \sum_{i=0}^{\infty} v_{(i+1)m}^{(r)} x^i = \sum_{j=1}^{r+1} js(r,m,j)(-x)^{j-1}$$

x^0: on the left, $s(2,m,0)v_m^{(2)} = v_m^{(2)}$ = right-hand side;

x^1: on the left, $-s(2,m,1)v_m^{(2)} + s(2,m,0)v_{2m}^{(2)} = \alpha_{21}^{2m} + \alpha_{22}^{2m} - (\alpha_{21}^m + \alpha_{22}^m)^2$
$= -2(\alpha_{21}\alpha_{22})^m,$
$= -2s(2,m,2)$
= right-hand side.

We note that

$$[(r + 2)/(j + 2)] = 0 \quad \text{for} \quad j > r \geq 0$$

and

$$r > [(r + 2)/(j + 2)] \quad \text{for} \quad 0 \leq j < r \quad \text{if} \quad r > 2,$$

where $[\cdot]$ represents the greatest integer function.

Theorem 1: The lacunary recurrence relation for $v_n^{(r)}$ for $r \geq 2$ is given by

$$\sum_{n=0}^{\min(r,j)} (-1)^n s(r,m,n) v^{(r)}_{(j-n+1)}$$

$$= (-1)^j (j + 1) s(r,m,j + 1)\ 1 - \delta\ r, [(r + 2)/(j + 2)] \quad \text{for positive } j.$$

Proof: We have from the lemma that

$$\sum_{n=0}^{\infty} (-1)^n s(r,m,n) x^n \sum_{i=0} v^{(r)}_{(i+1)m} x^i = \sum_{j=1}^{r+1} js(r,m,j)(-x)^{j-1}$$

which can be rearranged to give

$$\sum_{j=0}^{\infty} \sum_{n=0}^{j} (-1)^n s(r,m,n) v^{(r)}_{(j-n+1)m} x^j = \sum_{j=0}^{r} (j + 1) s(r,m,j + 1)(-x)^j.$$

On equating coefficients of x^j, we get

$$\sum_{n=0}^{j} (-1)^n s(r,m,n) v^{(r)}_{(j-n+1)m} = \begin{cases} 0 & \text{if } j > r, \\ (-1)^j (j + 1) s(r,m,j + 1) & \text{if } 0 \leq j \leq r. \end{cases}$$

But

$$\left(1 - \delta\left(r, [(r + 2)/(j + 2)]\right)\right) = \begin{cases} 0 & \text{for } j > r \\ 1 & \text{for } 0 \leq j < r,\ r > 2, \end{cases}$$

and $0 \leq n < r$ in $s(r,m,n)$ from which we get the required result when $r > 2$, as we exclude negative subscripts for $v_n^{(r)}$.

We next discuss the case for $r = 2$.

When j is unity, we get

$$s(r,m,0) v^{(r)}_{2m} - s(r,m,1) v^{(r)}_m = 2s(r,m,2)$$

which can be reorganized as

$$v^{(r)}_{2m} - (v^{(r)}_m)^2 + 2s(r,m,2) = 0.$$

When $r = 2$, this becomes

$$v^{(2)}_{2m} - (v^{(2)}_m)^2 + 2P^m_{22} = 0,$$

which is in agreement with Equation (3.16) of Horadam [2].

Similarly, when $j = 2$, we find that for arbitrary r,

$$s(r,m,0) v^{(r)}_{3m} - s(r,m,1) v^{(r)}_{2m} + s(r,m,2) v^{(r)}_m = 3s(r,m,4)$$

or

$$v^{(r)}_{3m} - v^{(r)}_m v^{(r)}_{2m} + s(r,m,2) v^{(r)}_m = 3s(r,m,4),$$

which, when $r = 2$, becomes

$$v_{3m}^{(2)} - v_m^{(2)} v_{2m}^{(2)} + P_{22}^m v_m^{(2)} = 0,$$

and this also agrees with Equation (3.16) of Horadam if we put $n = 2m$ and $w_m^{(2)} = v_m^{(2)}$ there. Thus, the theorem also applies when $r = 2$ if $j \geq 1$. If j were zero, and $r = 2$, since $\delta(2,[4/2]) = 1$, the theorem would reduce to

$$s(r,m,0)v_m^{(2)} = 0,$$

which is false.

Corollary 1: $v_{km}^{(r)} = \sum_{n=1}^{r} (-1)^{n+1} s(r,m,n) v_{(k-n)m}^{(r)}$.

Proof: Put $j = k - 1 > r$ in the theorem and we get

$$\sum_{n=0}^{r} (-1)^n s(r,m,n) v_{(k-n)m}^{(r)} = 0$$

which gives

$$\sum_{n=1}^{r} (-1)^{n+1} s(r,m,n) v_{(k-n)m}^{(r)} = v_{km}^{(r)} .$$

A particular case of the corollary occurs when $m = 1$, namely

$$v_k^{(r)} = \sum_{n=1}^{r} (-1)^{n+1} s(r,1,n) v_{k-n}^{(r)}$$
$$= \sum_{n=1}^{r} (-1)^{n+1} P_{rn} v_{k-n}^{(r)},$$

as we would expect.

The recurrence relation in Theorem 1 has gaps; for instance, there are missing numbers between $v_{(j+1)m}^{(r)}$ and $v_{jm}^{(r)}$. When $j = m = 2$, the lacunary recurrence relation becomes

$$v_6^{(r)} - s(r,2,1)v_4^{(r)} + s(r,2,2)v_2^{(r)} - s(r,2,3)v_0^{(r)}$$
$$= 3s(r,2,3)\big(1 - \delta(r,[(r + 2)/4])\big),$$

and the numbers $v_1^{(r)}$, $v_3^{(r)}$, and $v_5^{(r)}$ are missing. For further discussion of lacunary recurrence relations, see Lehmer [5]. The lacunary recurrence relations can be used to develop formulas for $v_n^{(r)}$.

3. GENERALIZED SEQUENCE

In this section we consider the more generalized sequence $\{w_n^{(r)}\}$.

Theorem 2: $w_{tn}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} s(r,t,j) w_{t(n-j)}^{(r)}, \quad n > r.$

Proof: Put

$$w_n^{(r)} = \sum_{j=1}^{r} A_j \alpha_{rj}^n$$

in which the A_j will be determined by the initial values of $\{w_{rj}^{(r)}\}$.

$$\sum_{j=1}^{r}(-1)^{j+1}s(r,t,j)w^{(r)}_{t(n-j)} = \sum_{j=1}^{r}(-1)^{j+1}s(r,t,j)\sum_{i=1}^{r}A_i\alpha_{ri}^{tn-t}$$

$$= \sum_{j=1}^{r}\alpha_{rj}^{t}\sum_{i=1}^{r}A_i\alpha_{ri}^{tn-t} - \sum_{\substack{j,\,k=1\\ j\neq k}}^{r}\alpha_{rj}^{t}\alpha_{rk}^{t}\sum_{i=1}^{r}A_i\alpha_{ri}^{tn-2t}$$

$$+ \cdots + (-1)^{r+1}(\alpha_{r1}^{t}\alpha_{r2}^{t}\ \ldots\ \alpha_{rr}^{t})\sum_{i=1}^{r}A_i\alpha_{ri}^{tn-rt}$$

$$= \sum_{j=1}^{r}A_j\alpha_{rj}^{tn} + \sum_{\substack{j,\,k=1\\ j\neq k}}^{r}A_j\alpha_{rj}^{tn-t}\alpha_{rk}^{t} - \sum_{\substack{j,\,k=1\\ j\neq k}}^{r}A_j\alpha_{rj}^{tn-t}\alpha_{rk}^{t}$$

$$- \sum_{\substack{i,\,j,\,k=1\\ i\neq j\neq k}}^{r}A_i\alpha_{ri}^{tn-2t}\alpha_{rj}^{t}\alpha_{rk}^{t} + \cdots$$

$$= \sum_{j=1}^{r}A_j\alpha_{rj}^{tn} = w_{tn}^{(r)},$$

as required.

When $t = r = 2$, we have $s(2,2,1) = 3$ and $s(2,2,2) = 1$, so that if $w_n^{(2)} = F_n$, the nth Fibonacci

$$F_{2n} = 3F_{2n-2} - F_{2n-4},$$

which result has been used by Rebman [8] and Hilton [2] in their combinatorial studies. There, too, the result

$$n = \sum_{\gamma(n)}(-1)^{k-1}F_{2a_1}F_{2a_2}\ \ldots\ F_{2a_k}$$

was useful.

[$\gamma(n)$ indicates summation over all compositions $(a_1, \ldots, a_k)$ of n, the number of components being variable.] The lacunary generalization of this result can be expressed as

Theorem 3: $W_n^{(r)} = \sum_{\gamma(n)}(-1)^{k-1}w_{ta_1}^{(r)}\ \ldots\ w_{ta_k}^{(r)}$, in which

$$W_n^{(r)} = \sum_{j=1}^{r}(-1)^{j+1}\{s(r,t,j) + h_j\}W_{j-n}^{(r)},\quad n > r,$$

where

$$h_j = \sum_{m=1}^{j}(-1)^m s(r,t,j-m)w_{tm}^{(r)}.$$

That the theorem generalizes the result can be seen if we let $r = 2$, $t = 1$, and $w_n^{(2)} = F_n$ again. Then, as before,

$$F_{2n} = 3F_{2n-2} - F_{2n-4}$$

and

$$W_n^{(2)} = \sum_{j=1}^{2}(-1)^{j+1}\{s(2,2,j) + h_j\}W_{n-j}^{(2)}$$

$$= \{s(2,2,2) + h_1\}W_{2-1}^{(2)} - \{s(2,2,2) + h_2\}W_{n-2}^{(2)}$$

$$= \{s(2,2,1) - s(2,2\ 0)F_2\}W_{n-1}^{(2)} - \{s(2,2,2) - s(2,2,1) + s(2,2,0)F_4\}W_{n-2}^{(2)}$$

$$= (3 - 1)_{n-1}^{(2)} - (1 - 3 + 3)W_{n-2}^{(2)} = 2W_{n-1}^{(2)} - W_{n-2}^{(2)};$$

i.e., $W^{(2)} = n$ as in the result.

To prove Theorem 3, we need the following lemmas.

Lemma 3.1: $W(x) = w(x)/(1 + w(x))$, where

$$W(x) = \sum_{n=1}^{\infty} W_n^{(r)} x^n \quad \text{and} \quad w(x) = \sum_{n=1}^{\infty} w_{tn}^{(r)} x^n.$$

Proof:

$$W(x) = \sum_{n=1}^{\infty} W_n^{(r)} x^n$$

$$= \sum_{n=1}^{\infty} \left(\sum_{\Upsilon(x)} (-1)^{k-1} w_{ta_1}^{(r)} \ldots w_{ta_k}^{(r)} \right) x^n$$

$$= \sum_{k=1}^{\infty} - \left(-\sum_{n=1}^{\infty} w_{tn}^{(r)} x^n \right)^k$$

$$= \sum_{k=1}^{\infty} - (-w(x))^k$$

$$= w(x)/(1 + w(x)).$$

Lemma 3.2: If $f(x) = \sum_{j=0}^{r} (-1)^{r-j} s(r,t,j) x^j$,

and

$$h(x) = \sum_{j=1}^{r} (-1)^{r-j} h_j x^j,$$

where

$$h(x) = f(x) w(x),$$

then

$$h_j = \sum_{m=1}^{j} (-1)^m s(r,t,j - m) w_{tm}^{(r)}.$$

Proof: If $h(x) = f(x)w(x)$,

then

$$W(x) = f(x)w(x)/(f(x) + f(x)w(x)) = h(x)/(f(x) + h(x)),$$

so that

$$h(x) = (f(x) + h(x))W(x).$$

Now

$$h(x) = \sum_{m=1}^{\infty} w_{tn}^{(r)} x^n \sum_{j=0}^{r} (-1)^{r-j} s(r,t,j) x^j$$

$$= \sum_{j=1}^{r} \left(\sum_{m=1}^{j} (-1)^{r-j+m} s(r,t;j - m) w_{tm}^{(r)} \right) x^j$$

$$+ \sum_{j=1}^{\infty} \left(\sum_{m=0}^{r} (-1)^m s(r,t,r - m) w_{(j+m)}^{(r)} \right) x^{r+j}$$

$$= \sum_{j=1}^{r} (-1)^{r-j} \left(\sum_{m=1}^{j} (-1) \;\; s(r,t,j-m) w^{(r)} \right) x^{j}$$

from Theorem 2. The result follows when the coefficients of x are equated. Thus,

$$f(x) + h(x) = \sum_{j=1}^{r} (-1)^{r-j} \{ s(r,t,j) + h_j \} x^j + 1.$$

And since

$$h(x) = \big(f(x) + h(x) \big) w(x),$$

Theorem 3 follows.

Shannon and Horadam [10] have looked at the development of second-order lacunary recurrence relations by using the process of multisection of series. The same approach could be used here. Riordan [9] treats the process in more detail.

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