ROOTS OF $(H - L)/15$ RECURRENCE EQUATIONS IN GENERALIZED PASCAL TRIANGLES

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1. INTRODUCTION

In this paper, we shall examine the roots of recurrence equations for $(H - L)/15$ sequences in Pascal's binomial, trinomial, quadrinomial, pentanomial, hexanomial, and heptanomial triangles.

Recall that the regular Lucas and Fibonacci sequences have the recurrence equation

$$x^2 - x - 1 = 0,$$

with roots

$$\alpha = (1 + \sqrt{5})/2 \quad \text{and} \quad \beta = (1 - \sqrt{5})/2.$$

As the roots of the $(H - L)/15$ sequences are examined, α and β appear frequently.

Generalized Pascal triangles arise from the multinomial coefficients obtained by the expansion of

$$(1 + x + x^2 + \cdots + x^{j-1})^n, \; j \geq 2, \; n \geq 0,$$

where n denotes the row in each triangle. For $j = 6$, the hexanomial coefficients give rise to the following triangle:

1															
1	1	1	1	1	1										
1	2	3	4	5	6	5	4	3	2	1					
1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1
etc.															

In order to explain the $(H - L)/15$ sequences, we shall first define sums of partition sets in the rows of Pascal triangles. The partition sums are defined

$$S(n,j,k,r) = \sum_{i=0}^{M} \left[\!\!\left[\begin{matrix} n \\ r + ik \end{matrix} \right]\!\!\right]_j ; \; 0 \leq r \leq k - 1, \quad M = \left[\frac{(j - 1)n - r}{k} \right],$$

the brackets denoting the greatest integer function. To clarify, we give a numerical example. Consider $S(3,6,15,0)$. This denotes the partition sums in the third row of the hexanomial triangle in which every fifteenth element is added, beginning with the zeroth column. The $S(3,6,15,0) = 1 + 3 = 4$. (Conventionally, the column of 1's at the far left is the zeroth column and the top row is the zeroth row.)

In the nth row of the j-nomial triangle the sum of the elements is j^n. This is expressed by

$$S(n,j,k,0) + S(n,j,k,1) + \cdots + S(n,j,k,k - 1) = j^n.$$

$$S(n,j,k,0) = (j^n + A_n)/k$$

$$S(n,j,k,1) = (j^n + B_n)/k \;\ldots$$

$$S(n,j,k,k - 1) = (j^n + Z_n)/k.$$

Since $S(0,j,k,0) = 1$,

$$S(0,j,k,1) = 0 \;\ldots\; S(0,j,k,k - 1) = 0,$$

we can solve for A_0, B_0, ..., Z_0 to get $A_0 = k - 1$, $B_0 = -1$, ..., $Z_0 = -1$.

Now a departure table can be formed with A_0, B_0, ..., Z_0 as the zeroth row. The term *departure* refers to the quantities, A_n, B_n, ..., Z_n that depart from the average value j^n/k. Pascal's rule of addition is the simple method for finding the successive rows in each departure table. The departure table for 15 partitions in the hexanomial triangle appears below. Each row has 15 elements which have been spread out by the computer into 3 *rows*.

TABLE 1

SUMS OF FIFTEEN PARTITIONS IN THE HEXANOMIAL TRIANGLE

14.	-1.	-1.	-1.	-1.
-1.	-1.	-1.	-1.	-1.
-1.	-1.	-1.	-1.	-1.
9.	9.	9.	9.	9.
9.	-6.	-6.	-6.	-6.
-6.	-6.	-6.	-6.	-6.
-21.	-6.	9.	24.	39.
54.	39.	24.	9.	-6.
-21.	-36.	-36.	-36.	-36.
-186.	-171.	-126.	-66.	9.
99.	159.	189.	189.	159.
99.	9.	-66.	-126.	-171.
-441.	-711.	-846.	-846.	-711.
-441.	-96.	264.	579.	804.
894.	804.	579.	264.	-96.
2004.	399.	-1251.	-2676.	-3651.
-3996.	-3651.	-2676.	-1251.	399.
2004.	3249.	3924.	3924.	3249.
18354.	16749.	12249.	5649.	-1926.
-9171.	-14826.	-17901.	-17901.	-14826.
-9171.	-1926.	5649.	12249.	16749.

The $(H - L)/15$ sequences are obtained from the difference of the maximum and minimum value sequences in a departure table, divided by 15, where 15 is the number of partitions. A table of $(H - L)/15$ sequences follows.

TABLE 2

$(H - L)/15$ SEQUENCES IN j-NOMIAL TRIANGLES

j =	2	3	4	5	6	7
	1	1	1	1	1	1
	1	1	1	1	1	1
	2	3	4	5	6	7
	3	7	12	19	25	28
	6	19	44	80	116	140
	10	51	153	331	528	658
	20	141	553	1379	2417	3164
	35	392	1960	5740	11053	15106
	70	1098	7042	23906	50562	72302

(continued)

j =	2	3	4	5	6	7
	126	3085	25080	99565	231283	345775
	252	8688	89861	414704	1057967	1654092
	462	24498	320661	1727341	4839483	7911790
	924	69136	1147444	7194890	22137392	37846314
	1716	195209	4098172	29969004	101263708	
	3431	551370		124831190	463213542	

2. BINOMIAL TRIANGLE

The pivotal element method was used to derive the $(H - L)/15$ recurrence equation in the binomial triangle,

$$x^7 - x^6 - 6x^5 + 5x^4 + 10x^3 - 6x^2 - 4x + 1 = 0.$$

We factor out $x - 1$ to get

$$x^6 - 6x^4 - x^3 + 9x^2 + 3x - 1 = 0,$$

which can be written as

$$x^2(x^2 - 3)^2 - x(x^2 - 3) - 1 = 0.$$

Let $y = x(x^2 - 3)$, then the equation above becomes

$$y^2 - y - 1 = 0,$$

with the roots α and β.

Solve $x(x^2 - 3) = \alpha$ and $x(x^2 - 3) = \beta$. β is the root of the first because

$$\beta(\beta^2 - 3) = \left(\frac{1 - \sqrt{5}}{2}\right)\left(\frac{6 - 2\sqrt{5}}{2} - 3\right) = \left(\frac{1 - \sqrt{5}}{2}\right)\left(\frac{-6 - 2\sqrt{5}}{4}\right)$$
$$= \frac{-6 + 4\sqrt{5} + 10}{8} = \frac{4 + 4\sqrt{5}}{8} = \frac{1 + \sqrt{5}}{2} = \alpha.$$

α is a root of the second because $\alpha(\alpha^2 - 3) = \beta$. We factor out $x - \beta$ from the first to get

$$x^2 + \beta x + (-3 + \beta^2) = 0,$$

and factor out $x - \alpha$ from the second to get

$$x^2 + \alpha x + (-3 + \alpha^2) = 0.$$

These quadratic equations have roots

$$\frac{-\beta \pm \sqrt{-3\beta^2 + 12}}{2} \quad \text{and} \quad \frac{-\alpha \pm \sqrt{-3\alpha^2 + 12}}{2}.$$

Thus the roots of the recurrence equation are

$$1,\ \alpha,\ \beta,\ \frac{-\beta \pm \sqrt{-3\beta^2 + 12}}{2},\ \frac{-\alpha \pm \sqrt{-3\alpha^2 + 12}}{2}.$$

3. TRINOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^6 - 6x^5 + 9x^4 + 5x^3 - 15x^2 + 5 = 0.$$

We rewrite as

$$(x^2(x - 3))^2 + 5x^2(x - 3) + 5 = 0.$$

Let $y = x^2(x - 3)$, then the equation above becomes

$$y^2 + 5y + 5 = 0,$$

with the roots $-\sqrt{5}\alpha$ and $\sqrt{5}\beta$.

Solve $x^2(x - 3) = -\sqrt{5}\alpha$ and $x^2(x - 3) = \sqrt{5}\beta$. α is a root of the first, since

$$\alpha^2(\alpha - 3) = \left(\frac{6 + 2\sqrt{5}}{4}\right)\left(\frac{1 + \sqrt{5}}{2} - 3\right) = \left(\frac{6 + 2\sqrt{5}}{4}\right)\left(\frac{-5 + \sqrt{5}}{2}\right)$$

$$= \frac{-20 - 4\sqrt{5}}{8} = -5\left(\frac{\sqrt{5} + 1}{2}\right) = -\sqrt{5}\alpha.$$

β is a root of the second, since $\beta^2(\beta - 3) = \sqrt{5}\beta$. We factor out $x - \alpha$ from the first to get

$$x^2 + (-3 + \alpha)x + (-3 + \alpha^2) = 0,$$

and factor out $x - \beta$ from the second to get

$$x^2 + (-3 + \beta)x + (-3 + \beta^2) = 0.$$

Since $-3 + \alpha = \sqrt{5}\beta$, and $-3 + \beta = -\sqrt{5}\alpha$, the roots to these quadratic equations may be simplified to

$$\frac{-\sqrt{5}\beta \pm \sqrt{3\sqrt{5}\alpha}}{2} \quad \text{and} \quad \frac{\sqrt{5}\alpha \pm \sqrt{3\sqrt{5}(-\beta)}}{2}.$$

Thus the roots of the recurrence equation again include α and β.

4. QUADRINOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^6 - x^5 - 10x^4 + 10x^2 + x - 1 = 0.$$

We factor out $(x - 1)(x + 1)$ to get

$$x^4 - x^3 - 9x^2 - x + 1 = 0.$$

Divide through by x^2, then let $y = x + 1/x$. We obtain

$$(2 + x^2 + 1/x^2) - (x + 1/x) - 11 = 0.$$

Then, after substituting y, the equation above becomes

$$y^2 - y - 11 = 0,$$

with the roots

$$\frac{1 \pm 3\sqrt{5}}{2}.$$

Now we solve

$$x + 1/x = \frac{1 \pm 3\sqrt{5}}{2}.$$

Multiply this equation by x to obtain

$$x^2 - \left(\frac{1 \pm 3\sqrt{5}}{2}\right)x + 1 = 0.$$

The roots of this pair of quadratic equations are found to be

$$\frac{1 \pm 3\sqrt{5}}{2} \pm \sqrt{\frac{1 + 45 \pm 6\sqrt{5}}{4} - 4} = \frac{1 \pm 3\sqrt{5} \pm 2\sqrt{3\sqrt{5}\alpha}}{4}.$$

Now $\dfrac{1 + 3\sqrt{5}}{2} = 2\alpha - \beta$ and $\dfrac{1 - 3\sqrt{5}}{2} = 2\beta - \alpha$, thus we may simplify. The roots of the recurrence equation are, therefore,

$$+1,\ -1,\ \frac{2\beta - \alpha \pm \sqrt{3\sqrt{5}(-\beta)}}{2},\ \frac{2\alpha - \beta \pm \sqrt{3\sqrt{5}\alpha}}{2}.$$

5. PENTANOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^5 - 5x^4 + 15x^2 - 9 = 0.$$

We factor out $x + 1$ to get

$$x^4 - 6x^3 + 6x^2 + 9x - 9 = 0.$$

Let $y = x - 3/2$. Then $y^2 = x - 3x + 9/4$ and

$$y^4 = x^4 - 6x^3 + (27/2)x^2 - (27/2)x + 81/16,$$

so the recurrence equation may be written

$$y^4 - (15/2)y^2 + (45/16) = 0.$$

Letting $z = y^2$ produces a quadratic equation in z with roots

$$\frac{\frac{15}{2} \pm \sqrt{\frac{225}{4} - 4\left(\frac{45}{16}\right)}}{2} = \frac{15 \pm 6\sqrt{5}}{4}.$$

Thus,

$$y = \frac{\pm\sqrt{15 \pm 6\sqrt{5}}}{2} \quad \text{and} \quad x = \frac{3 \pm \sqrt{15 \pm 6\sqrt{5}}}{2}.$$

We rewrite these last four roots as follows:

$$\frac{3 \pm \sqrt{15 + 6\sqrt{5}}}{2} = \frac{3 \pm \sqrt{3\sqrt{5}\alpha^3}}{2} \quad \text{and} \quad \frac{3 \pm \sqrt{15 - 6\sqrt{5}}}{2} = \frac{3 \pm \sqrt{3\sqrt{5}(-\beta^3)}}{2}.$$

Thus, the five roots to the recurrence equation are the four just above and -1.

6. HEXANOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^5 - 4x^4 - 5x^3 + 10x^2 + 5x - 5 = 0.$$

We factor out $x + 1$ to get

$$x^4 - 5x^3 + 10x - 5 = 0.$$

To use Ferrari's solution of the quartic equation, we must determine a, b, and k such that

$$x^4 - 5x^3 + 10x - 5 + (ax + b)^2 = (x^2 - (5/2)x + k)^2.$$

The determination of a, b, and k is accomplished by equating the coefficients of like powers of x in the equations above. This leads to the relations

$$a^2 = 2k + 25/4;\ 2ab + 10 = -5k;\ b^2 - 5 = k^2$$

which gives rise to the resolvent cubic equation in k:

$$8k^3 - 60k + 25 = 0.$$

A root of this cubic equation is $k = 5/2$. We substitute this value of k in the relations above to solve for a and b. We find

$$a = \frac{3\sqrt{5}}{2} \quad \text{and} \quad b = \frac{-3\sqrt{5}}{2}.$$

Now we solve an equation in which both members are perfect squares:

$$(x^2 - (5/2)x + (5/2))^2 = (ax + b)^2.$$

Therefore,

$$x^2 - (5/2)x + (5/2) = ax + b$$

and

$$x^2 - (5/2)x + (5/2) = -ax - b.$$

The four roots of the quartic equation can be found by solving these two quadratic equations. We substitute the values of a and b in these quadratic equations to obtain

$$x^2 - \left(\frac{5 + 3\sqrt{5}}{2}\right)x + \frac{5 + 3\sqrt{5}}{2} = 0$$

and

$$x^2 - \left(\frac{5 - 3\sqrt{5}}{2}\right)x + \frac{5 - 3\sqrt{5}}{2} = 0.$$

Hence

$$x = \frac{\frac{5 \pm 3\sqrt{5}}{2} \pm \sqrt{\frac{70 \pm 30\sqrt{5}}{4} - 4\left(\frac{5 \pm 3\sqrt{5}}{2}\right)}}{2} = \frac{5 \pm 3\sqrt{5} \pm \sqrt{30 \pm 6\sqrt{5}}}{4}.$$

Thus, $$x = \frac{\sqrt{5}\alpha^2 \pm \sqrt{3}\sqrt{\sqrt{5}}\sqrt{\alpha}}{2} \quad \text{or} \quad x = \frac{-\sqrt{5}\beta^2 \pm \sqrt{3}\sqrt{\sqrt{5}}\sqrt{-\beta}}{2}.$$

These four roots together with $x = -1$ comprise the five roots to the recurrence equation.

7. HEPTANOMIAL TRIANGLE

We derived the $(H - L)/15$ recurrence equation

$$x^7 - 4x^6 - 6x^5 + 10x^4 + 5x^3 - 6x^2 - x + 1 = 0.$$

Let $y = 1/x$ to obtain

$$1 - 4y - 6y^2 + 10y^3 + 5y^4 - 6y^5 - y^6 + y^7 = 0.$$

This equation in y is precisely the recurrence equation of the $(H - L)/15$ sequence in the binomial triangle. Hence, the roots we are seeking are the reciprocals of the roots that were derived in the binomial triangle. These reciprocal roots are

$$1,\ 1/\alpha,\ 1/\beta,\ \frac{\beta^3 \pm \sqrt{3\sqrt{5}(-\beta^3)}}{2},\ \frac{\alpha^3 \pm \sqrt{3\sqrt{5}\alpha^3}}{2}.$$

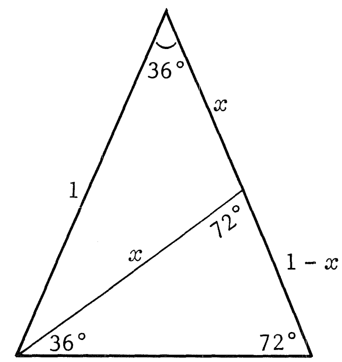
8. CONCLUSION

How surprising to see α and β appear with such frequency in the roots to all the cases with 15 partitions!

Another unexpected result was the reciprocal relationship that occurred between the recurrence equations of the binomial and heptanomial triangles. More study into the successive j-nomial triangles could certainly surface more interesting results.

Theorem A: $\cos 12° = \dfrac{-\beta + \sqrt{3}\sqrt{\sqrt{5}}\sqrt{\alpha}}{4}$.

Proof: Consider the Golden Triangle:



$x = 2 \sin 18°$

Therefore $\dfrac{1 - x}{x} = \dfrac{x}{1}$, which implies $x = \dfrac{\sqrt{5} - 1}{2} = -\beta$.

$$\cos 18° = 1 - \left(\frac{3 - \sqrt{5}}{8}\right) = \frac{\sqrt{5}}{4}\left(\frac{1 + \sqrt{5}}{2}\right)$$

$$\cos 18° = \frac{\sqrt{\sqrt{5}}}{2}\sqrt{\alpha}$$

$$\sin 18° = \frac{\sqrt{5} - 1}{4}$$

$$\begin{aligned}\cos 12° &= \cos 18° \cos 30° + \sin 18° \sin 30°\\ &= \frac{\sqrt{\sqrt{5}}}{2}\sqrt{\alpha}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(-\frac{\beta}{2}\right) = \frac{-\beta + \sqrt{3}\sqrt{\sqrt{5}}\sqrt{\alpha}}{4}.\end{aligned}$$

We note this occurs often in the roots.

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