# POWERS OF THE PERIOD FUNCTION FOR THE SEQUENCE OF FIBONACCI NUMBERS

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If $m$ is an integer greater than or equal to 2, we write $\phi(m)$ for the length of the period of the sequence of Fibonacci numbers reduced to least nonnegative residues modulo $m$. The function $\phi$ has been studied quite extensively (see, for example, [1], [2], and [3]). It is easy to discover that for small values of $m$ there exists a positive integer $k$ such that

$$\phi^k(m) = \phi^{k+1}(m),$$

i.e., that the sequence

$$\phi(m),\ \phi(\phi(m)),\ \phi(\phi(\phi(m))),\ \ldots$$

eventually becomes stationary. The purpose of this note is to prove this fact in general.

We start by observing that it is sufficient to consider $m$ to be of the form $2^a3^b5^c$ for nonnegative integers $a$, $b$, and $c$. For, if $\psi(m)$ denotes the rank of apparition of $m$ in the Fibonacci sequence modulo $m$, then by Lemma 12 of [1], if $p \neq 5$ is an odd prime we have $\psi(p)\,|\,(p \pm 1)$, while $\psi(5) = 5$. Thus, for an odd prime $q \neq 5$ with $q \geq p$ such that $q\,|\,\psi(p)$, we have that $q\,|\,(p \pm 1)$, which is impossible. Consequently, the primes occurring in the prime decomposition of $\psi(p)$ are all less than $p$ or, as we shall say, $\psi(p)$ "involves" only primes less than $p$. Now, by a Theorem of Vinson [2], we know that

$$\phi(p) = 2^r\psi(p) \text{ where } r = 0,\ 1,\ \text{or } 2,$$

so that $\phi(p)$ also involves only primes less than $p$.

Suppose $\phi(m) = dp^\beta$, where $p$ is a prime greater than 5, and $d$ involves only primes less than $p$ and $\beta \neq 0$. Then using Lemma 14 of [1] and Theorem 5 of [3] we have that

$$\phi^2(m) = \begin{cases} [\phi(d),\ p^{\beta-1}\phi(p)] & \text{if } \phi(p^2) \neq \phi(p) \\ [\phi(d),\ p^{\beta-2}\phi(p)] & \text{if } \phi(p^2) = \phi(p) \text{ and } \beta \neq 1 \end{cases}$$

where square brackets with integers inside denote the lowest common multiple of those integers. Now, $\phi(d)$ and $\phi(p)$ involve only primes less than $p$, so that $\phi^2(m) = d_1p^\gamma$, say, where $0 \leq \gamma < \beta$ and $d_1$ involves only primes less than $p$. Carrying on in this way, we eventually find an integer $s$ such that $\phi^s(m)$ does not involve $p$ and so, continuing, we may find an integer $t$ such that $\phi^t(m)$ involves only 2, 3, and 5. Thus

$$\phi^t(m) = 2^a3^b4^c \text{ for some } a,\ b,\ c \geq 0.$$

This justifies the assertion that we need consider only integers of the stated form.

We now define a sequence $\{\alpha_n\}$ by $\alpha_1 = a - 1$, where $a > 1$, and $\alpha_{n+1} = \max(\alpha_n - 1,\ 3)$ if $n \geq 1$. Then it is easy to see that $\{\alpha_n\}$ eventually takes the constant value 3: in fact, $\alpha_{a-3} = 3$ if $a \geq 5$ and $\alpha_2 = 3$ if $a < 5$. Now $\phi^n(2^a) = 2^{\alpha_n} \cdot 3$, so that if $a \geq 5$ we have $\phi^{a-3}(2^a) = 2^3 \cdot 3$, and if $a < 5$ we have $\phi^2(2^a) = 2^3 \cdot 3$. Thus, we see that there exists an integer $u \geq 2$ such that $\phi^u(2^a) = 2^3 \cdot 3$ if $a > 1$. Similarly, if we define the sequence $\{\beta_n\}$ by

$\beta_1 = b - 1$, where $b > 1$, and $\beta_{n+1} = \max(\beta_n - 1, 1)$ if $n \geq 1$, we have that $\beta_{b-1} = 1$ if $b \geq 3$, $\beta_2 = 1$ if $b < 3$, and that $\phi^n(3^b) = 2^3 \cdot 3^{\beta_n}$. Thus, there exists an integer $v \geq 2$ such that $\phi^v(3^b) = 2^3 \cdot 3$ if $b > 1$.

Now we note that $\phi^4(2) = \phi^3(3) = 2^3 \cdot 3$ and that $\phi^3(5^c) = 2^3 \cdot 3 \cdot 5^c$ for any $c \geq 1$ and that $\phi(2^3 \cdot 3 \cdot 5^c) = 2^3 \cdot 3 \cdot 5^c$ holds even for $c = 0$. Again using Lemma 14 of [1] we have for $a, b > 1$ that

$$\begin{aligned}\phi^{u+v}(2^a 3^b) &= [\phi^{u+v}(2^a),\ \phi^{u+v}(3^b)]\\ &= [\phi^v(2^3 \cdot 3),\ \phi^u(2^3 \cdot 3)]\\ &= 2^3 \cdot 3,\end{aligned}$$

so that

$$\phi^{u+v}(2^a 3^b 5^c) = [2^3 \cdot 3,\ 2^3 \cdot 3 \cdot 5^c] = 2^3 \cdot 3 \cdot 5^c$$

since $u + v > 3$. Consequently

$$\phi^{u+v+1}(2^a 3^b 5^c) = \phi^{u+v}(2^a 3^b 5^c).$$

The remaining cases are when $a \leq 1$ or $b \leq 1$, and it is easy to check that $\phi^{v+3}(2^a 3^b 5^c) = \phi^{v+2}(2^a 3^b 5^c)$ if $a \leq 1$ and $\phi^{u+3}(2^a 3^b 5^c) = \phi^{u+2}(2^a 3^b 5^c)$ if $b \leq 1$.

## REFERENCES

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2. J. Vinson. "The Relation of the Period Modulo $m$ to the Rank of Apparition of $m$ in the Fibonacci Sequence." *The Fibonacci Quarterly* 1, No. 1 (1963):37-45.
3. D. D. Wall. "Fibonacci Series Modulo $m$." *American Math. Monthly* 67 (1960):525-532.

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# SOME REMARKS ON THE PERIODICITY OF THE SEQUENCE OF FIBONACCI NUMBERS—II

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The Fibonacci sequence $\{F_n\}$ is defined by

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \quad (n \geq 1).$$

If $t$ is an integer greater than 2 and $\phi(t)$ is the length of the period of the sequence reduced to least nonnegative residues modulo $t$, it was shown in [2] that $\phi(F_{m-1} + F_{m+1}) = 4m$ if $m$ is even and $\phi(F_{m-1} + F_{m+1}) = 2m$ if $m$ is odd. It follows for $m > 4$ that

$$\phi(F_{m-1} + F_{m+1}) = \frac{1}{2}(\phi(F_{m-1}) + \phi(F_{m+1})).$$

I conjectured in the same paper that if $m - k > 3$ then

$$\phi(F_{m-k} + F_{m+k}) = \frac{k}{2}(\phi(F_{m-k}) + \phi(F_{m+k})).$$

The object of this note is to show that this conjecture is false and to give the correct answer in some special cases.