ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within 2 months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-313 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA*

A. Show that the Fibonacci numbers partition the Fibonacci numbers.
B. Show that the Lucas numbers partition the Fibonacci numbers.
(See "Additive Partitions I," FQJ, April 1977, p. 166.)

H-314 *Proposed by P. Bruckman, Concord, CA*

Given $x_0 \in (-1, 0)$, define the sequence $S = (x_n)_{n=0}^{\infty}$ as follows:

(1) $$x_{n+1} = 1 + (-1)^n\sqrt{1 + x_n}, \quad n = 0, 1, 2, \ldots .$$

Find the limit point(s) of S, if any exist.

H-315 *Proposed by D. P. Laurie, National Research Institute for Mathematical Sciences, Pretoria, South Africa*

Let the polynomial P be given by

$$P(z) = z_n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1z + a_0$$

and let $z_1, z_2, \ldots, z_n$ be distinct complex numbers. The following iteration scheme for factorizing P has been suggested by Kerner [1]:

$$\hat{z}_i = z_i - \frac{P(z_i)}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n}(z_i - z_j)}; \quad i = 1, 2, \ldots, n.$$

Prove that if $\sum\limits_{i=1}^{n} z_i = -a_{n-1}$, then also $\sum\limits_{i=1}^{n} \hat{z}_i = -a_{n-1}$.

REFERENCE

1. I. Kerner. "Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen." *Numer. Math.* 8 (1966):290-294.

H-316 *Proposed by B. R. Myers, University of British Columbia, Vancouver, Canada*

The enumerator of compositions with exactly k parts is $(x + x^2 + \cdots)^k$, so that

$$(1) \qquad [W(x)]^k = (w_1x + w_2x^2 + \cdots)^k$$

is then the enumerator of weighted k-part compositions. After Hoggatt & Lind ("Compositions and Fibonacci Numbers," *The Fibonacci Quarterly* 7 (1969):253-266), the number of weighted compositions of n can be expressed in the form,

$$(2) \qquad C_n(w) = \sum_{\gamma(n)} w_{a_1} \ldots w_{a_k} \qquad (n > 0),$$

where $w = \{w_1, w_2, \ldots\}$ and where the sum is over all compositions $a_1 + \cdots + a_k$ of n (k variable). In particular (*ibid.*),

$$(3) \qquad \sum_{\gamma(n)} a_1 \ldots a_k = F_{2n}(1, 1),$$

where $F_k(p, q)$ is the kth number in the Fibonacci sequence

$$(4) \qquad \begin{aligned} F_1(p, q) &= p \quad (\geq 0) \\ F_2(p, q) &= q \quad (\geq p) \\ F_{n+2}(p, q) &= F_{n+1}(p, q) + F_n(p, q) \quad (n \geq 1). \end{aligned}$$

Show that

$$(5) \qquad \sum_{\gamma(n)} (a_1 \pm 1)a_1 \ldots a_k = 2[F_{2n\pm1}(1, 1) - 1]$$

and, hence, that

$$(6) \qquad \sum_{\gamma(n)} (a_1 - 1)a_1 \ldots a_k + \sum_{\gamma(n)} a_1 \ldots a_k = F_{2n}(1, 1 + 2m) - 2m \quad (m \geq 0).$$

SOLUTIONS

Umbral - a

H-285 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA*
(A generalization of R. G. Buschman's H-18)
(Vol. 16, No. 2, April 1978)

Show that

$$(a) \qquad \sum_{k=0}^{n} \binom{n}{k} F_{rk} L_{nr-rk} = 2^n F_{rn} \text{ or } (F^r + L^r)^n \doteq (2F^r)^n.$$

Solution by L. Carlitz, Duke University, Durham, NC

Much more can be proved readily. Let $C(n, k)$, $0 \leq k \leq n$, be numbers that satisfy the symmetry condition:

$$C(n, k) = C(n, n - k) \qquad (0 \leq k \leq n).$$

Let a, b be arbitrary, and define

$$F_n = \frac{a^n - b^n}{a - b}, \quad L_n = a^n + b^n.$$

Then

$$\sum_{k=0}^{n} C(n, k) F_{rk} L_{rn-rk} = \frac{1}{a - b} \sum_{k=0}^{n} C(n, k)(a^{rk} - b^{rk})(a^{rn-rk} + b^{rn-rk})$$

$$= \frac{1}{a - b} \sum_{k=0}^{n} C(n, k)(a^{rn} - b^{rn}) + \frac{1}{a - b} \sum_{k=0}^{n} C(n, k)(a^{rk}b^{rn-rk} - a^{rn-rk}b^{rk}).$$

Since

$$\sum_{k=0}^{n} C(n, k)a^{rn-rk}b^{rk} = \sum_{k=0}^{n} C(n, n-k)a^{rk}b^{rn-rk} = \sum_{k=0}^{n} C(n, k)a^{rk}b^{rn-rk},$$

it follows that

$$\sum_{k=0}^{n} C(n, k)(a^{rk}b^{rn-rk} - a^{rn-rk}b^{rk}) = 0.$$

Therefore

$$(*) \qquad \sum_{k=0}^{n} C(n, k)F_{rk}L_{rn-rk} = F_{rn}\sum_{k=0}^{n} C(n, k).$$

For example, if $C(n, k) = \binom{n}{k}$, we get

$$\sum_{k=0}^{n}\binom{n}{k}F_{rk}L_{rn-rk} = 2\,F_{rn},$$

while, if $C(n, k) = \binom{n}{k}^2$, we have

$$\sum_{k=0}^{n}\binom{n}{k}^2 F_{rk}L_{rn-rk} = \binom{2n}{n}F_{rn}.$$

To take a less obvious example, let $A_{n,k}$ denote the Eulerian number defined by

$$\sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}; \quad A_n(x) = \sum_{k=1}^{n} A_{n,k}x^k \quad (n \geq 1).$$

It is well known that

$$A_{n,k} = A_{n,n-k} \quad (1 \leq k \leq n)$$

and

$$\sum_{k=1}^{n} A_{n,k} = n!$$

Take

$$C(n, k) = A_{n+1,k+1} \; (0 \leq k \leq n),$$

so that

$$C(n, k) = C(n, n-k) \qquad (0 \leq k \leq n).$$

It follows that

$$\sum_{k=0}^{n} A_{n+1,k+1}F_{rk}L_{rn-rk} = (n+1)!F_{rn}.$$

Also solved by P. Bruckman, J. Vogel, and the proposer.

LATE ACKNOWLEDGMENTS:

H-281, also solved by J. Shallit.
H-283, also solved by A. Shannon, A. Philippou, and P. Yff.

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