5. *continued*

$$t_{29} + t_{69} = t_{75}, \quad t_{168} - t_{69} = t_{153},$$
$$t_{29}t_{69} = t_{1449}, \quad t_{168}t_{69} = t_{8280}.$$

6. For the system of equations,

(16) $$t_x + t_y = t_u, \qquad t_x t_y = t_v,$$

there exists also the solution:

$$t_{505} + t_{531} = t_{733}, \quad t_{505}t_{531} = t_{189980}.$$

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ON EULER'S SOLUTION TO A PROBLEM OF DIOPHANTUS—II

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1. INTRODUCTION

In an earlier paper [1] we considered solutions to a system of equations:

$$x_i x_j + 1 = y_{ij}^2; \quad 1 \le i < j \le n.$$

In this note we look at the generalized problems:

(1.1) $$x_i x_j + a = y_{ij}^2, \quad a \ne 0.$$

In Section 2 we apply the results of [1] to the solutions of (1.1). In Section 3 we consider the following problem: Find $n \times 2$ matrices

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

so that $a_i b_j \pm a_j b_i = \pm 1$ for all $1 \le i < j \le n$. In Section 4 we apply the results of Section 3 to get two-parameter families of solutions of (1.1), linear in a, for $n = 4$.

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2. SOLUTIONS

Solutions of

$$x_i x_3 + a = y_{i3}^2; \quad i = 1, 2,$$

where

$$x_3, y_{i3} \in R = k[x_1, x_2, \sqrt{x_1 x_2 + a}]$$

and k is a field of characteristic $\neq 2$; x_1, x_2 algebraically independent over k.

We saw in [1] that for $a = 1$ the general solution could be represented by

$$(2.1) \quad \sqrt{x_1}y_{23} + \sqrt{x_2}y_{13} = \pm(\sqrt{x_1} \pm \sqrt{x_2})(y_{12} + \sqrt{x_1 x_2})^n; \; n = 0, \pm 1, \pm 2, \ldots .$$

where $y_{12} = \sqrt{x_1 x_2 + a}$. We arrived at (2.1) by solving the Pell's equation,

$$(2.2) \quad x_1 y_{23}^2 - x_2 y_{13}^2 = x_1 - x_2,$$

which arises from the elimination of x_3 between the two equations (1.1). For general a, equation (2.2) becomes

$$(2.3) \quad x_1 y_{23}^2 - x_2 y_{13}^2 = a(x_1 - x_2).$$

If a is a square in b, say $a = b^2$, then the solution of (2.3) is entirely analogous to (2.1).

Theorem (2.4): If $a = b^2$, then the general solution of (2.3) in R is given by

$$\sqrt{x_1}y_{23} + \sqrt{x_2}y_{13} = \pm b(\sqrt{x_1} \pm \sqrt{x_2})\left(\frac{y_{12} + \sqrt{x_1 x_2}}{b}\right)^n; \; n = 0, \pm 1, \pm 2, \ldots .$$

Proof: We just take the general solution (2.1) for the case $a = 1$ and rename x_i by x_i/b and y_{ij} by y_{ij}/b to get the solution for $a = b^2$.

In case a is not a square in k, we can use Theorem 2.4 to give the general solution in the extended ring $R^* = k^*[x_1, x_2, y_{12}]$ where $k^* = k(\sqrt{a})$. The solutions in R are therefore given by the following.

Theorem (2.5): If a is not a square in k, then the general solution of (2.3) in R is given by

$$\sqrt{x_1}y_{23} + \sqrt{x_2}y_{13} = \pm(\sqrt{x_1} \pm \sqrt{x_2})(y_{12} \pm \sqrt{x_1 x_2})^{2n+1}a^{-n}; \; n = 0, 1, 2, \ldots .$$

For example, if $k = 0$ and a is an integer, then either $a = \pm 1$ or the only solution with integral coefficients is

$$(2.6) \quad x_3 = x_1 + x_2 + 2y_{12}, \; y_{i3} = x_i + y_{12}.$$

Following [1], we see that in case $a = b^2$ we can find

$$x_4, y_{i4} \in R_1 = k[x_1, x_2, x_3, y_{12}, y_{13}, y_{23}]$$

so that $x_i x_4 + a = y_{i4}^2$. Namely,

$$(2.7) \quad x_4 = x_1 + x_2 + x_3 + 2\frac{x_1 x_2 x_3}{a} + 2\frac{y_{12}y_{13}y_{23}}{a}$$

$$y_{i4} = \frac{1}{b}(x_i y_{jk} + y_{ij} y_{ik}); \; \{i, j, k\} = \{1, 2, 3\}.$$

If a is not a square, then there is no x_4 element in R_1 so that $x_i x_4 + a$ are squares in R_1 for $i = 1, 2, 3$.

The construction in [1] for an $x_5 \in K = k(x_1, x_2, x_3, y_{12}, y_{13}, y_{23})$ so that $x_i x_5 + a = y_{i5}^2$; $i = 1, 2, 3, 4$ can be extended in case $a = b^2$ but not if a is not a square in k.

3. ON REAL $n \times 2$ MATRICES SATISFYING $a_i b_j \pm a_j b_i = \pm 1$

If we first consider the case where all the 2×2 determinants are ±1, then it is clear that we must have $n \leq 3$, since for $n = 4$ the 6 determinants A_{ij} satisfy the identity

$$A_{12}A_{34} + A_{31}A_{24} + A_{23}A_{14} = 0$$

which makes it impossible that all A_{ij} are odd integers. Of course, there are many solutions for $n = 3$, for example

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

There is no restriction on the size of the matrix if we require only that the permanents of the 2×2 submatrices are ±1. In fact, given any a, b so that $2ab = \pm 1$, then the matrix

$$a_1 = a_2 + \cdots + a_n = a; \quad b_1 = \cdots = b_n = b$$

obviously has all permanents ±1.

If we call a matrix *admissible* when it satisfies $a_i b_j \pm a_j b_i = \pm 1$ for all $1 \leq i < j \leq n$, then admissibility is preserved under the following operations.

(i) Change of sign of any element.
(ii) Interchange of the two rows and permutations of columns.
(iii) Multiplication of one row by any nonzero constant and division of the other row by the same constant.

We therefore normalize to consider only matrices with nonnegative entries and without repeated columns. We call such matrices *permissible*.

Lemma (3.1): A permissible matrix with an entry 0 has no more than three columns.

Proof: We normalize the matrix so that $a_1 = 1$, $b_1 = 0$. Then

$$b_2 = \cdots = b_n = 1.$$

Thus, if we order the columns by $a_2 \leq a_3 \leq \cdots \leq a_n$, we get $a_j \pm a_i = 1$ for $2 \leq i < j \leq n$. If $n > 3$, this leaves only the possibilities

$$a_3 = 1 - a_2, \quad a_4 = 1 + a_2.$$

But then, $a_4 + a_3 = 2$ and $a_4 - a_3 = 2a_2 = 1$ leads to $a_2 = a_3 = 1/2$. Thus, $n \leq 3$.

We then assume that all entries are positive, and normalize to the form

$$\begin{pmatrix} 1 & a_2 & \cdots & a_n \\ b & b_2 & \cdots & b_n \end{pmatrix} \text{ with } 1 \leq a_2 \leq \cdots \leq a_n.$$

Then $b_i = 1 + ba_i$ or $|1 - ba_i|$.

Case 1. $b_2 = 1 + ba_2$. From the equations

$$a_2|1 \pm ba_i| \pm (1 + ba_2)a_i = \pm 1,$$

we get three possibilities:

$$a_2(1 + ba_i) - a_i(1 + ba_2) = -1, \qquad a_i = a_2 + 1$$

or

$$a_2(1 - ba_i) - a_i(1 + ba_2) = -1, \qquad a_i = \frac{a_2 + 1}{1 + 2ba_2}$$

or

$$a_2(ba_i - 1) + a_i(1 + ba_2) = 1, \qquad a_i = \frac{a_2 + 1}{1 + 2ba_2}.$$

Thus, $n \leq 4$, and for $n = 4$ we have

$$a_3 = \frac{a_2 + 1}{1 + 2ba_2}, \qquad b_3 = \frac{1 - b + ba_2}{1 + 2ba_2};$$

$$a_4 = a_2 + 1, \qquad b_4 = 1 + b + ba_2.$$

The equation $a_3b_4 \pm a_4b_3 = \pm 1$ becomes

$$(a_2 + 1)[(1 + b + ba_2) \pm (1 - b + ba_2)] = 1 + 2ba_2,$$

and hence,

$$2(a_2 + 1)(1 + ba_2) = 1 + 2ba_2,$$

which is impossible, or

$$2b(a_2 + 1) = 1 + 2ba_2, \quad b = 1/2.$$

But then $a_3 = 1$, $b_3 = 1/2$ which is not permissible. Thus $n \leq 3$ in this case.

Case 2. $b_2 = 1 - ba_2$. We get the possibilities:

$$(3.1) \qquad \begin{aligned} a_2(1 + ba_i) - (1 - ba_2)a_i &= \pm 1, & a_i &= \frac{a_2 \pm 1}{1 - 2ba_2} \\ a_2(1 - ba_i) + (1 - ba_2)a_i &= 1, & a_i &= \frac{a_2 - 1}{2ba_2 - 1} \\ a_2(1 - ba_i) - (1 - ba_2)a_i &= -1, & a_i &= a_2 + 1 \\ a_2(ba_i - 1) + (1 - ba_2)a_i &= 1, & a_i &= a_2 + 1 \\ a_2(ba_i - 1) - (1 - ba_2)a_i &= \pm 1, & a_i &= \frac{a_2 \pm 1}{2ba_2 - 1} \end{aligned}$$

So the possible coices of a_i, $i = 3, 4, \ldots$, depend on the magnitude of ba_2.

(i) For $ba_2 < 1/2$, we get the possibilities:

$$(3.2) \qquad \begin{aligned} a_i &= \frac{a_2 - 1}{1 - 2ba_2}, & b_i &= \frac{1 - b - ba_2}{1 - 2ba_2}; \\ a_i &= \frac{a_2 + 1}{1 - 2ba_2} & b_i &= \frac{1 + b - ba_2}{1 - 2ba_2}; \\ a_i &= a_2 + 1, & b_i &= 1 - b - ba_2. \end{aligned}$$

(ii) For $1/2 = ba_2$, we get only one possibility

$$a_i = a_2 + 1, \quad b_i = 1 - b - ba_2.$$

(iii) For $1/2 < ba_2 < 1$, we get the possibilities:

$$(3.2)' \quad \begin{aligned} a_i &= \frac{a_2 - 1}{2ba_2 - 1}, & b_i &= \frac{|1 - b - ba_2|}{2ba_2 - 1} \\ a_i &= \frac{a_2 + 1}{2ba_2 - 1}, & b_i &= \frac{1 + b - ba_2}{2ba_2 - 1} \\ a_i &= a_2 + 1, & b_i &= |1 - b - ba_2|. \end{aligned}$$

The first and third lines in (3.2) lead to

$$(1 - b - ba_2)[(a_2 + 1) \pm (a_2 - 1)] = 1 - 2ba_2;$$

that is, either

$$2a_2(1 - b - ba_2) = 1 - 2ba_2 \quad \text{or} \quad 2a_2(1 - ba_2) = 1,$$

which is impossible, since $a_2 > 1$ and $1 - ba_2 > 1/2$; or

$$2(1 - b - ba_2) = 1 - 2ba_2 \quad \text{or} \quad b = 1/2,$$

which violates the condition $ba_2 < 1/2$.

The second and third lines in (3.2) lead to

$$(a_2 + 1)[1 + b - ba_2 \pm (1 - b - ba_2)] = 1 - 2ba_2;$$

that is, either

$$2(a_2 + 1)(1 - ba_2) = 1 - 2ba_2 \quad \text{or} \quad a_i = \frac{a_2 + 1}{1 - 2ba_2} = \frac{1}{2(1 - ba_2)} < 1,$$

contrary to hypothesis, or

$$(3.3) \qquad 2b(a_2 + 1) = 1 - 2ba_2$$

$$b = \frac{1}{2(2a_2 + 1)}$$

which yields the 4 x 2 matrix

$$(3.4) \qquad \begin{pmatrix} 1 & a & a + 1 & 2a + 1 \\ \frac{1}{4a + 2} & \frac{3a + 2}{4a + 2} & \frac{3a + 1}{4a + 2} & \frac{3}{2} \end{pmatrix}$$

where the parameter, a, is chosen ≥ 1.

The first and second lines of (3.2) lead to

$$(a_2 + 1)(1 - b - ba_2) \pm (a_2 - 1)(1 + b - ba_2) = \pm(1 - 2ba_2)^2$$

which gives

$$(2b + 1)(2ba_2^2 - 2a_2 + 1) = 0 \quad \text{or} \quad 2(1 - 2ba_2) = (1 - 2ba_2)^2.$$

The first violates $2ba_2 < 1$, and the second violates $2ba_2 > 0$. Thus, (3.4) is the only matrix with $n > 3$ for Case 2(i).

The second and third lines of (3.2)′ lead to

$$(a_2 + 1)[1 + b - ba_2 \pm (1 - b - ba_2)] = 2ba_2 - 1.$$

Thus, either

$$2(a_2 + 1)(1 - ba_2) = 2ba_2 - 1, \quad b = \frac{2a_2 + 3}{2a_2(a_2 + 2)},$$

or

$$2b(a_2 + 1) = 2ba_2 - 1,$$

which is impossible.

The first case leads to the matrix

$$(3.4') \qquad \begin{pmatrix} 1 & a & a + 1 & a + 2 \\ \frac{2a + 3}{2a(a + 2)} & \frac{1}{2a(a + 2)} & \frac{a + 3}{2a(a + 2)} & \frac{3}{2a} \end{pmatrix}$$

This is the same as the matrix (3.4) in case $0 < a \leq 1$, after we renormalize by replacing a by $1/a$, multiplying the first row by a and the second row by $1/a$ and interchanging the first two columns.

The first and third lines of (3.2) lead to

$$|1 - b - ba_2|[(a_2 + 1) \pm (a_2 - 1)] = 2ba_2 - 1,$$

both of which lead to

$$a_i = \frac{|1 - b - ba_2|}{2ba_2 - 1} \leq \frac{1}{2} < 1,$$

contrary to hypothesis.

To consider the first and third lines we first note that the conditions $1 - b - ba_2 < 0$, that is,

$$b > 1/(1 + a_2)$$

and

$$a_i = (a_2 - 1)/(2ba_2 - 1) \geq a_2 \geq 1$$

and incompatible. Thus, we get

$$(a_2 + 1)(1 - b - ba_2) \pm (a_2 - 1)(1 + b - ba_2) = (2ba_2 - 1)^2,$$

which leads either to

$$2a_2(1 - ba_2) - 2b = (2ba_2 - 1)^2,$$

and hence,

$$2(1 - b - ba_2) \leq (2ba_2 - 1)^2, \quad a_i \leq \frac{1}{2};$$

or to $ab_2 = \frac{1}{2}$. Both cases are excluded.

Thus (3.4) is the only normalized 4×2 matrix in Case 2.

Case 3. $b_2 = ba_2 - 1$. In this case, $b_i = ba_i - 1$ for all i and the possibilities reduce to:

$$(3.5) \qquad \begin{aligned} a_2(ba_i - 1) - a_i(ba_2 - 1) = 1, &\quad a_i = a_2 + 1, \\ a_2(ba_i - 1) + a_i(ba_2 - 1) = 1, &\quad a_i = \frac{a_2 + 1}{2ba_2 - 1}. \end{aligned}$$

The two lines of (3.5) lead to

$$(a_2 + 1)[(ba_2 + b - 1) \pm (-ba_2 + b + 1)] = 2ba_2 - 1.$$

The resulting equations are $2b(a_2 + 1) = 2ba_2 - 1$, which is impossible, and

$$b = \frac{2a_2 + 1}{2a_2^2}$$

which makes

$$a_3 = \frac{a_2 + 1}{2ba_2 - 1} = a_2.$$

To sum up.

Theorem (3.6): There are no 5 x 2 permissible real matrices, and there is a one-parameter family of normalized permissible 4 x 2 matrices, given by (3.4).

We have limited the discussion to real matrices in order to reduce the number of cases. However, the family of permissible matrices (3.4) is valid for all fields of characteristic $\neq 2$ or 3, as long as we exclude the values $a = 0, -1/3, -1/2, -2/3$, and -1.

4. PARAMETRIC SOLUTIONS OF (1.1) WITH THE USE OF ADMISSIBLE MATRICES

Theorem (4.1): Given an admissible matrix $\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix}$ then for any a, the elements

$$x_i = a_i^2 a - b_i^2; \quad i = 1, 2, \dots, n,$$

satisfy (1.1) with $y_{ij} = a_i a_j a \pm b_i b_j$.

Proof: For $1 \leq i < j \leq n$, we have

$$x_i x_j + a = (a_i^2 a - b_i^2)(a_j^2 a - b_j^2) + a \tag{4.2}$$

$$= a_i^2 a_j^2 a^2 + (1 - a_i^2 b_j^2 - a_j^2 b_i^2)a^2 + b_i^2 b_j^2.$$

Now, since $a_i b_j \pm a_j b_i = \pm 1$, we have

$$1 - a_i^2 b_j^2 - a_j^2 b_i^2 = \pm 2a_i a_j b_i b_j.$$

Substituting in (4.2), we get

$$x_i x_j + a = a_i^2 a_j^2 a^2 \pm 2a_i a_j b_i b_j a + b_i^2 b_j^2 = (a_i a_j a \pm b_i b_j)^2.$$

In view of (3.4), we get a two-parameter family of 4 x 2 admissible matrices,

$$\begin{pmatrix} s & st & s(t+1) & s(2t+1) \\ \frac{1}{2s(2t+1)} & \frac{3\quad + 2}{2s(2t+1)} & \frac{3\quad + 1}{2s(2t+1)} & \frac{3}{2s} \end{pmatrix}$$

which yield a corresponding three-parameter solution,

$$x_i = x_i(s, t, a), \quad y_{ij} = y_{ij}(s, t, a),$$

of (1.1), which is linear in a. In general, x_3 and x_4 are algebraic, but not rational, functions of x_1 and x_2.

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