REFERENCES

1. J. Alanen. "Empirical Study of Aliquot Series." (Unpublished doctoral dissertation, Yale University, 1972.)
2. P. Cattaneo. "Sui numeri quasiperfetti." *Boll. Un. Mat. Ital.* (3) 6 (1951):59-62.
3. L. E. Dickson. "Finiteness of the Odd Perfect and Primitive Abundant Numbers With n Distinct Factors." *Amer. J. Math.* 35 (1913):413-422.
4. P. Hagis, Jr. "Every odd Perfect Number Has at Least Eight Distinct Prime Factors." *Notices, Amer. Math. Society* 22, No. 1 (1975):Ab. 720-10-14.
5. P. Hagis, Jr. "A Lower Bound for the Set of Odd Perfect Numbers." *Math. Comp.* 27, No. 124 (1973).
6. M. Kishore. "Quasiperfect Numbers Have at Least Six Distinct Prime Factors." *Notices, Amer. Math. Society* 22, No. 4 (1975):Ab. 75T-A113.
7. C. Pomerance. "On the Congruences $\sigma(n) \equiv a \pmod{n}$ and $n \equiv a \pmod{\phi(n)}$." *Acta Arith.* 26 (1974):265-272,

*****

WEIGHTED STIRLING NUMBERS OF THE FIRST AND SECOND KIND—I

L. Carlitz
Duke University, Durham, N.C. 27706

1. INTRODUCTION

The Stirling numbers of the first and second kind can be defined by

$$(x)_n \equiv x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} S_1(n, k)x^k \tag{1.1}$$

and

$$x^n = \sum_{k=0}^{n} S(n, k)x(x-1)\cdots(x-k+1), \tag{1.2}$$

respectively.

It is well known that $S_1(n, k)$ is the number of permutations of

$$Z_n = \{1, 2, \ldots, n\}$$

with k cycles and that $S(n,k)$ is the number of partitions of the set Z_n into k blocks [1, Ch. 5], [2, Ch. 4]. These combinatorial interpretations suggest the following extensions.

Let n, k be positive integers, $n \geq k$, and let $k_1, k_2, \ldots, k$ be nonnegative integers such that

$$\begin{cases} k = k_1 + k_2 + \cdots + k_n \\ n = k_1 + 2k_2 + \cdots + nk_n. \end{cases} \tag{1.3}$$

We define $\overline{S}(n, k, \lambda)$, $\overline{S}_1(n, k, \lambda)$, where λ is a parameter, in the following way.

$$\overline{S}(n, k, \lambda) = \sum\sum(k_1\lambda + k_2\lambda^2 + \cdots + k_n\lambda^n), \tag{1.4}$$

where the inner summation is over all partitions of Z_n into k_1 blocks of cardinality 1, k_2 blocks of cardinality 2, ..., k_n blocks of cardinality n; the outer summation is over all $k_1, k_2, \ldots, k_n$ satisfying (1.3).

(1.5) $$\overline{S}_1(n,\ k,\ \lambda) = \sum\sum\left\{k_1(\lambda)_1 + k_2\frac{(\lambda)_2}{1!} + \cdots + k_n\frac{(\lambda)}{(n-1)!}\right\},$$

where the inner summation is over all permutations of Z_n with k_1 cycles of length 1, k_2 cycles of length 2, ..., k_n cycles of length n; the outer summation is over all $k_1,\ k_2,\ \ldots,\ k_n$ satisfying (1.3).

We now put

(1.6) $$\begin{cases} S(n,\ k,\ \lambda) = \frac{1}{k}\overline{S}\,(n,\ k,\ \lambda) \\ S_1(n,\ k,\ \lambda) = \frac{1}{n}\overline{S}_1\,(n,\ k,\ \lambda). \end{cases}$$

It is evident from (1.4) and (1.5) that

(1.7) $$S(n,\ k,\ 1) = S(n,\ k),\ S_1(n,\ k,\ 1) = S_1(n,\ k).$$

Indeed we shall show that if λ is an integer, then $S(n,\ k,\ \lambda)$ and $S_1(n,\ k,\ \lambda)$ are also integers. More precisely, we show that, for arbitrary λ,

(1.8) $$\overline{S}(n,\ k,\ \lambda) = \sum_{j=1}^{n-k+1}(k)_j S(n,\ j+k-1)\binom{\lambda}{j},$$

(1.9) $$\overline{S}_1(n,\ k,\ \lambda) = \sum_{j=1}^{n-k+1}\binom{n}{j}(\lambda)_j S_1(n-j,\ k-1).$$

We obtain recurrences and generating functions for both $S(n,\ k,\ \lambda)$ and $S_1(n,\ k,\ \lambda)$. Simpler results hold for the functions

(1.10) $$\begin{cases} R(n,\ k,\ \lambda) = \overline{S}(n,\ k+1,\ \lambda) + S(n,\ k) \\ R_1(n,\ k,\ \lambda) = \overline{S}_1(n,\ k+1,\ \lambda) + S_1(n,\ k). \end{cases}$$

For example, we have the recurrences

(1.11) $$\begin{cases} R(n+1,\ k,\ \lambda) = R(n,\ k-1,\ \lambda) + (k+\lambda)R(n,\ k,\ \lambda) \\ R_1(n+1,\ k,\ \lambda) = R_1(n,\ k-1,\ \lambda) + (n+\lambda)R_1(n,\ k,\ \lambda) \end{cases}$$

and the orthogonality relations

(1.12) $$\sum_{j=0}^{n} R(n,\ j,\ \lambda)\cdot(-1)^{j-k}R_1(j,\ k,\ \lambda)$$

$$= \sum_{j=0}^{n}(-1)^{n-j}R_1(n,\ j,\ \lambda)R(j,\ k,\ \lambda) = \begin{cases} 1 & (n=k) \\ 0 & (n \neq k). \end{cases}$$

For $\lambda = 0$ and $\lambda = 1$, (1.11) and (1.12) reduce to familiar formulas for $S(n,\ k)$ and $S_1(n,\ k)$.

The definitions (1.4) and (1.5) furnish combinatorial interpretations of $\overline{S}(n,\ k,\ \lambda)$ and $\overline{S}_1(n,\ k,\ \lambda)$ when λ is arbitrary. For λ a nonnegative integer, the recurrences (1.11) suggest combinatorial interpretations for $R(n,\ k,\ \lambda)$ and $R_1(n,\ k,\ \lambda)$ that generalize the interpretation of $S(n,\ k)$ and $S_1(n,\ k)$ described above. For the statement of the generalized interpretations, see Section 7 below.

2. THE FUNCTION $\overline{S}(n, k, \lambda)$

Let n, k be positive integers, $n \geq k$, and $k_1, k_2, \ldots, k_n$ nonnegative such that

$$
\left\{ \begin{aligned} k &= k_1 + k_2 + \cdots + k_n \\ n &= k_1 + 2k_2 + \cdots + nk_n. \end{aligned} \right. \tag{2.1}
$$

Put

$$
S(n;\, k_1,\, k_2,\, \ldots,\, k_n;\, \lambda) = \sum (k_1\lambda + k_2\lambda^2 + \cdots + k_n\lambda^n), \tag{2.2}
$$

where the summation is over all partitions of $Z_n = 1, 2, \ldots, n$ into k_1 blocks of cardinality 1, k_2 blocks of cardinality 2, ..., k_n blocks of cardinality n. Then we have (compare [2, p. 75]):

$$
\sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k_1, k_2, \ldots} S(n;\, k_1,\, k_2,\, \ldots;\, \lambda) \frac{y_1^{k_1} y_2^{k_2} \cdots}{k_1! k_2! \cdots}
$$

$$
= \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k_1, k_2, \ldots} (k_1\lambda + k_2\lambda^2 + \cdots) \frac{n!}{1!^{k_1} 2!^{k_2} \cdots} \frac{y_1^{k_1} y_2^{k_2} \cdots}{k_1! k_2! \cdots}
$$

$$
= \left(\frac{y_1 \lambda x}{1!} + \frac{y_2 \lambda^2 x^2}{2!} + \cdots \right) \exp\left\{ \frac{y_1 x}{1!} + \frac{y_2 x^2}{2!} + \cdots \right\}.
$$

For $y_1 = y_2 = \cdots = y$, the extreme right member becomes

$$
y(e^{\lambda x} - 1) \exp\{y(e^x - 1)\}.
$$

Hence, we get the generating function

$$
\sum_{n,k} \overline{S}(n, k, \lambda) \frac{x^n}{n!} y^k = y(e^{\lambda x} - 1) \exp\{y(e^x - 1)\}. \tag{2.3}
$$

Recall that

$$
\sum_{n,k} S(n, k) \frac{x^n}{n!} y^k = \exp\{y(e^x - 1)\}. \tag{2.4}
$$

Thus, the right-hand side of (2.3) is equal to

$$
y \sum_{m=1}^{\infty} \frac{\lambda^m x^m}{m!} \sum_{n,k} S(n, k) \frac{x^n}{n!} y^k
$$

and therefore,

$$
\overline{S}(n, k, \lambda) = \sum_{m=1}^{n-k+1} \binom{n}{m} \lambda^m S(n - m, k - 1). \tag{2.5}
$$

Note that, for $\lambda = 1$, (2.3) reduces to

$$
\sum_{n,k} \overline{S}(n, k, 1) \frac{x^n}{n!} y^k = y(e \quad - 1) \exp\{y(e^x - 1)\} = y \frac{\partial}{\partial y} \exp\{y(e^x - 1)\}
$$

$$
= \sum_{n,k} kS(n, k) \frac{x^n}{n!} y^k, \quad \text{by (2.4)}.
$$

Thus, we again get

$$\bar{S}(n, k, 1) = kS(n, k).$$

By (1.2),

$$\lambda^m = \sum_{j=0}^{m} S(m, j)j!\binom{\lambda}{j}.$$

Thus, (2.5) becomes

$$\begin{aligned}\bar{S}(n, k, \lambda) &= \sum_{m=1}^{n-k+1}\binom{n}{m}S(n - m, k - 1)\sum_{j=1}^{m}S(m, j)j!\binom{\lambda}{j}\\ &= \sum_{j=1}^{n-k+1} j!\binom{\lambda}{j}\sum_{m=j}^{n}\binom{n}{m}S(m, j)S(n - m, k - 1).\end{aligned}$$

The inner sum is equal to

$$\binom{j + k - 1}{j}S(n, j + k - 1),$$

so that

$$\bar{S}(n, k, \lambda) = \sum_{j=1}^{n-k+1} j!\binom{\lambda}{j}\binom{j + k - 1}{j}S(n, j + k - 1) \tag{2.6}$$
$$= \sum_{j=1}^{n-k+1} (k)_j S(n, j + k - 1)\binom{\lambda}{j}.$$

Hence,

$$S(n, k, \lambda) = \frac{1}{k}\bar{S}(n, k, \lambda) = \sum_{j=1}^{n-k+1} (k + 1)_{j-1}S(n, j + k - 1)\binom{\lambda}{j}. \tag{2.7}$$

Thus, for λ an integer, $S(n, k, \lambda)$ is an integer. For example, we have

$$S(n, k, 1) = S(n, k)$$
$$S(n, k, 2) = 2S(n, k) + (k + 1)S(n, k + 1)$$
$$S(n, k, 3) = 3S(n, k) + 3(k + 1)S(n, k + 2).$$

It follows readily from (2.7) that

$$\sum_{t=0}^{m}(-1)^t\binom{m}{t}S(n, k, \lambda - t) \tag{2.8}$$
$$= \sum_{j=m}^{n-k+1} (k + 1)_{j-1}S(n, j + k - 1)\binom{\lambda - m}{j - m}, \quad (m \geq 1).$$

This result holds for all λ. However, if λ is a positive integer, then

$$\sum_{t=0}^{\lambda} (-1)^t\binom{\lambda}{t}S(n, k, \lambda - t) = (k + 1)_{\lambda-1}S(n, \lambda + k - 1), \tag{2.9}$$

and

$$(2.10)\quad \sum_{t=0}^{\lambda+1}(-1)^t\binom{\lambda+1}{t}S(n,\,k,\,\lambda-t) = \sum_{j=\lambda+1}^{n-k+1}(-1)^{j-\lambda-1}(k+1)_{j-1}S(n,\,j+k-1).$$

3. THE FUNCTION $R(n, k, \lambda)$

It is convenient to define

$$(3.1)\quad R(n,\,k,\,\lambda) = \overline{S}(n,\,k+1,\,\lambda) + S(n,\,k).$$

Thus, (2.5) implies

$$(3.2)\quad R(n,\,k,\,\lambda) = \sum_{m=0}^{n-k}\binom{n}{m}\lambda^m S(n-m,\,k),$$

while (2.7) gives

$$(3.3)\quad R(n,\,k,\,\lambda) = \sum_{j=0}^{n-k}(k+1)_j\,S(n,\,j+k)\binom{\lambda}{j}.$$

Multiplying (3.2) by $k!\binom{y}{k}$ and summing over k, we get

$$\sum_{k=0}^{n}k!\binom{y}{k}R(n,\,k,\,\lambda) = \sum_{m=0}^{n}\binom{n}{m}\lambda^m\sum_{k=0}^{n-m}S(n-m,\,k)y(y-1)\cdots(y-k+1)$$

$$= \sum_{m=0}^{n}\binom{n}{m}\lambda^m y^{n-m}.$$

Hence,

$$(3.4)\quad \sum_{k=0}^{n}k!\binom{y}{k}R(n,\,k,\,\lambda) = (y+\lambda)^n.$$

It follows from (3.4) that

$$(3.5)\quad \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{n}k!\binom{y}{k}R(n,\,k,\,\lambda) = e^{x(y+\lambda)}.$$

To obtain a recurrence for $R(n, k, \lambda)$, take

$$\sum_{k=0}^{n}k!\binom{y}{k}(R(n+1,\,k,\,\lambda) - \lambda R(n,\,k,\,\lambda)) = (y+\lambda)^{n+1} - \lambda(y+\lambda)^n$$

$$= y(y+\lambda)^n.$$

Since

$$k!\binom{y}{k}y = (k+1)!\binom{y}{k+1} + k.k!\binom{y}{k},$$

it is clear that (3.4) gives

$$R(n+1,\,k,\,\lambda) - \lambda R(n,\,k,\,\lambda) = kR(n,\,k,\,\lambda) + R(n,\,k-1,\,\lambda),$$

that is

$$R(n+1, k, \lambda) = (\lambda + k)R(n, k, \lambda) + R(n, k-1, \lambda). \tag{3.6}$$

An equivalent result is

$$\overline{S}(n+1, k+1, \lambda) = (\lambda + k)\overline{S}(n, k+1, \lambda) + \overline{S}(n, k, \lambda) + S(n, k). \tag{3.7}$$

To get an explicit formula for $R(n, k, \lambda)$ we recall that

$$S(n, k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

Thus, by (3.2),

$$R(n, k, \lambda) = \frac{1}{k!}\sum_{m=0}^{n-k}\binom{n}{m}\lambda^m\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^{n-m}.$$

For $n - k < m \le n$, the inner sum vanishes, so that

$$R(n, k, \lambda) = \frac{1}{k!}\sum_{m=0}^{n}\binom{n}{m}\lambda^m\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^{n-m}$$

$$= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{m=0}^{n}\binom{n}{m}\lambda^m j^{n-m}.$$

Thus,

$$R(n, k, \lambda) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\lambda + j)^n = \frac{1}{k!}\Delta^k\lambda^n. \tag{3.8}$$

It follows from (3.8) that

$$\sum_{n=k}^{\infty}R(n, k, \lambda)\frac{z^n}{n!} = \frac{1}{k!}e^{\lambda z}(e^z - 1)^k \tag{3.9}$$

in agreement with previous results. Also, since

$$\frac{1}{k!}\sum_{n=0}^{\infty}z^n\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\lambda + j)^n = \frac{1}{k!}\sum_{j=0}^{k}\frac{(-1)^{k-j}\binom{k}{j}}{1 - (\lambda + j)z}$$

$$= \frac{z^k}{(1 - \lambda z)(1 - (\lambda + 1)z)\ \ldots\ (1 - (\lambda + k)z)},$$

we have

$$\sum_{n=0}^{\infty}R(n, k, \lambda)z^n = \frac{z^k}{(1 - \lambda z)(1 - (\lambda + 1)z)\ \ldots\ (1 - (\lambda + k)z)}. \tag{3.10}$$

We also note that (3.9) implies the "addition theorem":

$$R(n, j + k, \lambda + \mu) = \binom{j+k}{j}^{-1}\sum_{m=0}^{n}\binom{n}{m}R(m, j, \lambda)R(n - m, k, \mu). \tag{3.11}$$

By the recurrence (3.6) together with $R(0, 0, \lambda) = 1$, or by means of (3.8), we have

(3.12) $$R(n, 0, \lambda) = \lambda^n, \quad R(n, n, \lambda) = 1.$$

Moreover, if we put

$$x^n = \sum_{k=0}^{n} \overline{R}(n, k, \lambda)(x - \lambda)(x - \lambda - 1) \cdots (x - \lambda - k + 1),$$

then

$$\overline{R}(n + 1, k, \lambda) = (\lambda + k)\overline{R}(n, k, \lambda) + \overline{R}(n, k - 1, \lambda),$$

so that $\overline{R}(n, k, \lambda) = R(n, k, \lambda)$. Thus, we have

(3.13) $$y^n = \sum_{k=0}^{n} R(n, k, \lambda)(y - \lambda)(y - \lambda - 1) \cdots (y - \lambda - k + 1),$$

or, replacing y by $-y$,

(3.14) $$y^n = \sum_{k=0}^{n} (-1)^{n-k} R(n, k, \lambda)(y + \lambda)_k.$$

This, of course, is equivalent to (3.4).

It is clear from (3.8) or (3.13) that

(3.15) $$R(n, k, 0) = S(n, k).$$

For $\lambda = 1$, since $\overline{S}(n, k, 1) = kS(n, k)$, then by (3.1)

$$R(n, k, 1) = (k + 1)S(n, k + 1) + S(n, k),$$

so that

(3.16) $$R(n, k, 1) = S(n + 1, k + 1).$$

The function

(3.17) $$B(n, \lambda) = \sum_{k=0}^{n} R(n, k, \lambda)$$

evidently reduces, for $\lambda = 0$, to the Bell number [1, p. 210]

$$B(n) = \sum_{k=0}^{n} S(n, k).$$

A few formulas may be noted. It follows from (3.2) that

(3.18) $$B(n, \lambda) = \sum_{m=0}^{n} \binom{n}{m} \lambda^m B(n - m).$$

Also, by (3.9), we have

(3.19) $$\sum_{n=0}^{\infty} B(n, \lambda)\frac{z^n}{n!} = e^{\lambda z} \exp(e^z - 1),$$

which, indeed, is implied by (3.18).

Differentiation of (3.19) gives

$$\sum_{n=0}^{\infty} B(n+1, \lambda)\frac{z^n}{n!} = \lambda e^{\lambda z} \exp(e^z - 1) + e^{(\lambda+1)z}\exp(e^z - 1).$$

Hence,

$$B(n+1, \lambda) = \lambda B(n, \lambda) + B(n, \lambda+1) \tag{3.20}$$

$$= B(n, \lambda) + \sum_{m=0}^{n} \binom{n}{m} B(m, \lambda).$$

Iteration of the first half of (3.20) gives

$$B(n+m, \lambda) = \sum_{j=0}^{m} \frac{1}{j!}\Delta^j \lambda^m \cdot B(n, \lambda+j), \tag{3.21}$$

as can be proved by induction on m. Incidentally, by (3.8), (3.21) can be written in the form

$$B(n+m, \lambda) = \sum_{j=0}^{m} R(m, j, \lambda)B(n, \lambda+j). \tag{3.22}$$

To anticipate the first result in Section 6, the inverse of (3.22) is

$$B(n, \lambda+m) = \sum_{j=0}^{m} (-1)^{m-j} R_1(m, j, \lambda)B(n+j, \lambda), \tag{3.23}$$

where $R_1(m, j, \lambda)$ is defined by (5.1).

*
* *

Returning to (3.9), note that

$$\sum_{n=k}^{\infty} R(n, k, \lambda+1)\frac{z^n}{n!} = \frac{1}{k!}e^{(\lambda+1)z}(e^z - 1)^k$$

$$= \frac{1}{k!}e^{\lambda z}(e^z - 1)^{k+1} + \frac{1}{k!}e^{\lambda z}(e^z - 1)^k,$$

which implies

$$R(n, k, \lambda+1) = (k+1)R(n, k+1, \lambda) + R(n, k, \lambda). \tag{3.24}$$

More generally, since

$$e^{mz} = ((e^z - 1) + 1)^m = \sum_{j=0}^{m} \binom{m}{j} (e^z - 1)^j,$$

we get

$$R(n, k, \lambda+m) = \sum_{j=0}^{m} \binom{m}{j} (k+1)_j R(n, k+j, \lambda). \tag{3.25}$$

We may also write (3.24) in the form

$$\Delta_\lambda R(n,\ k,\ \lambda) = (k+1)R(n,\ k+1,\ \lambda), \tag{3.26}$$

where Δ_λ is the finite difference operator. Iteration of (3.26) gives

$$\Delta_\lambda^m R(n,\ k,\ \lambda) = (k+1)_m R(n,\ k+m,\ \lambda). \tag{3.27}$$

4. THE FUNCTION $\overline{S}_1(n,\ k,\ \lambda)$

Corresponding to (2.2), we define

$$S_1(n;\ k_1,\ k_2,\ \ldots,\ k_n;\ \lambda) = k_1(\lambda)_1 + k_2\frac{(\lambda)_2}{1!} + \cdots + k_n\frac{(\lambda)_n}{(n-1)!}, \tag{4.1}$$

where the inner summation is over all permutations of Z_n,

$$n = k_1 + 2k_2 + \cdots + nk_n,$$

with k_1 cycles of length 1, k_2 cycles of length 2, ..., k_n cycles of length n. Then (compare [2, p. 68]), we have

$$\sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{k_1,k_2,\ldots} S_1(n;\ k_1,k_2,\ldots,k_n;\lambda)\frac{y_1^{k_1}y_2^{k_2}\cdots}{k_1!k_2!\cdots}$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{k_1,k_2,\ldots} k_1(\lambda)_1 + k_2\frac{(\lambda)_2}{1!} + \cdots + k_n\frac{(\lambda)_n}{(n-1)!}\left\{\frac{n!}{1^{k_1}2^{k_2}\cdots n^{k_n}}\right\}\frac{y_1^{k_1}y_2^{k_2}\cdots}{k_1!k_2!\cdots}$$

$$=\left\{\frac{(\lambda)_1}{1!}y_1x + \frac{(\lambda)_2}{2!}y_2x^2 + \frac{(\lambda)_3}{3!}y_3x^3 + \cdots\right\}\exp\left\{y_1x + \frac{1}{2}y_2x^2 + \frac{1}{3}y_3x^3 + \cdots\right\}.$$

For $y_1 = y_2 = \cdots y$, the extreme right member becomes

$$y((1-x)^{-\lambda} - 1)(1-x)^{-y}.$$

Hence, we get

$$\sum_{n,k}\overline{S}_1(n,\ k,\ \lambda)\frac{x^n}{n!}y^k = y((1-x)^{-\lambda} - 1)(1-x)^{-y}, \tag{4.2}$$

where

$$\overline{S}_1(n,\ k,\ \lambda) = \sum S_1(n;\ k_1,\ k_2,\ \ldots,\ k_n;\lambda), \tag{4.3}$$

and the summation on the right is over all nonnegative $k_1,\ k_2,\ \ldots,\ k_n$ satisfying $n = k_1 + 2k_2 + \cdots + nk_n$.

Since (see [2, p. 71]),

$$\sum_{n,k} S_1(n,\ k)\frac{x^n}{n!}y^k = (1-x)^{-y}, \tag{4.4}$$

it follows from (4.2) that

$$\sum_{n,k}\overline{S}_1(n,\ k+1,\ \lambda)\frac{x^n}{n!}y^k = \sum_{n,k} S_1(n,\ m)\frac{x^n}{n!}((\lambda+y)^m - y^m)$$

$$= \sum_{n,m} S_1(n,\ m)\frac{x^n}{n!}\sum_{k=0}^{m-1}\binom{m}{k}\lambda^{m-k}y^k = \sum_{n,k}\frac{x^n}{n!}y^k\sum_{m=k+1}^{n}\binom{m}{k}\lambda^{m-k}S_1(n,\ m).$$

Therefore,

$$\overline{S}_1(n,\ k+1,\ \lambda) = \sum_{j=1}^{n-k} \binom{j+k}{j} \lambda^j S_1(n,\ j+k). \tag{4.5}$$

In the next place, it also follows from (4.2) that

$$\sum_{n,k} \overline{S}_1(n,\ k+1,\ \lambda)\frac{x^n}{n!}y^k = ((1-x)^{-\lambda} - 1)(1-x)^{-y}$$

$$= \sum_{m=1}^{\infty} (\lambda)_m \frac{x^m}{m!} \sum_{n,k} S_1(n,\ k)\frac{x^n}{n!}y^k.$$

Equating coefficients, we get

$$\overline{S}_1(n,\ k+1,\ \lambda) = \sum_{m=1}^{n-k} \binom{n}{m} (\lambda)_m S_1(n-m,\ k)$$

$$= \sum_{m=1}^{n-k} \frac{(\lambda)_m}{m!} n(n-1) \cdots (n-m+1) S_1(n-m,\ k).$$

Thus,

$$S_1(n,\ k+1,\ \lambda) = \frac{1}{n}\overline{S}_1(n,\ k+1,\ \lambda) \tag{4.7}$$

$$= \sum_{m=1}^{n-k} \frac{(\lambda)_m}{m!} (n-1) \cdots (n-m+1) S_1(n-m,\ k).$$

It follows at once from (4.7) that, for λ integral, $S_1(n,\ k+1,\ \lambda)$ is also integral.

It is evident from (4.1) and (4.3) that

$$\overline{S}_1(n,\ k,\ 1) = nS_1(n,\ k). \tag{4.8}$$

Thus, for example, (4.5) and (4.6) yield

$$\sum_{j=1}^{n-k} \binom{j+k}{j} S_1(n,\ j+k) = nS_1(n,\ k+1), \tag{4.9}$$

and

$$\sum_{m=1}^{n-k} n(n-1) \cdots (n-m+1) S_1(n-m,\ k) = nS_1(n,\ k+1), \tag{4.10}$$

respectively.

5. THE FUNCTION $R_1(n,\ k,\ \lambda)$

We define the function $R_1(n,\ k,\ \lambda)$ by means of

$$R_1(n,\ k,\ \lambda) = \overline{S}_1(n,\ k+1,\ \lambda) + S_1(n,\ k). \tag{5.1}$$

Then, by (4.5),

$$R_1(n,\ k,\ \lambda) = \sum_{j=0}^{n-k} \binom{j+k}{j} \lambda^j S_1(n,\ j+k), \tag{5.2}$$

and by (4.6),

$$R_1(n,\ k,\ \lambda) = \sum_{m=0}^{n-k}\binom{n}{m}(\lambda)_m S_1(n - m,\ k) \tag{5.3}$$
$$= \sum_{m=0}^{n-k}\frac{(\lambda)_m}{m!}n(n-1)\ \cdots\ (n - m + 1)S_1(n - m,\ k).$$

It is also evident from (4.2) and (4.4) that

$$\sum_{n,k} R_1(n,\ k,\ \lambda)\frac{x^n}{n!}y^k = (1 - x)^{-\lambda - y}. \tag{5.4}$$

Differentiation of (5.4) with respect to x gives

$$\sum_{n,k} R_1(n + 1,\ k,\ \lambda)\frac{x^n}{n!}y^k = (\lambda + y)(1 - x)^{-\lambda - y - 1},$$

so that

$$(1 - x)\sum_{n,k} R_1(n+1,\ k,\ \lambda)\frac{x^n}{n!}y^k = (\lambda + y)\sum_{n,k} R_1(n,\ k,\ \lambda)\frac{x^n}{n!}y^k.$$

Equating coefficients, we get

$$R_1(n + 1,\ k,\ \lambda) = nR_1(n,\ k,\ \lambda) = \lambda R_1(n,\ k,\ \lambda) + R_1(n,\ k = 1,\ \lambda),$$

that is,

$$R_1(n + 1,\ k,\ \lambda) = (\lambda + n)R_1(n,\ k,\ \lambda) + R_1(n,\ k - 1,\ \lambda). \tag{5.5}$$

It follows at once from (5.5) and $R_1(0,\ 0,\ \lambda) = 1$ that

$$R_1(n,\ 0,\ \lambda) = (\lambda)_n,\quad R_1(n,\ n\lambda) = 1. \tag{5.6}$$

Also, taking $y = 1$ in (5.4), we get

$$\sum_{k=0}^{n} R_1(n,\ k,\ \lambda) = (\lambda + 1)_n. \tag{5.7}$$

More generally, we have

$$\sum_{k=0}^{n} R_1(n,\ k,\ \lambda)y^k = (\lambda + y)_n. \tag{5.8}$$

Clearly, (5.5) is implied by (5.8).

It is clear from (5.4) that

$$R_1(n,\ k,\ 0) = S_1(n,\ k). \tag{5.9}$$

For $\lambda = 1$, we have, by (4.8) and (5.1),

$$R_1(n,\ k,\ 1) = S_1(n + 1,\ k + 1). \tag{5.10}$$

These formulas may be compared with (3.15) and (3.16).

In view of (5.10), (5.2) and (5.3) reduce to

$$S_1(n + 1,\ k + 1) = \sum_{j=0}^{n-k}\binom{j + k}{j}S_1(n,\ j + k), \tag{5.11}$$

and

(5.12) $$S_1(n+1,\ k+1) = \sum_{m=0}^{n-k} n(n-1)\cdots(n-m+1)S_1(n-m,\ k).$$

It is not difficult to give direct proofs of (5.11) and (5.12).

Returning to (5.4), note that

$$(1-x)\sum_{n,k} R_1(n,\ k,\ \lambda+1)\frac{x^n}{n!}y^k = (1-x)^{-\lambda-y}.$$

This gives

(5.13) $$R_1(n,\ k,\ \lambda) = R_1(n,\ k,\ \lambda+1) - nR_1(n-1,\ k,\ \lambda+1),$$

and generally,

(5.14) $$R_1(n,\ k,\ \lambda) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}n(n-1)\cdots(n-j+1)R_1(n-j,\ k,\ \lambda+m).$$

The inverse of (5.14) is

(5.15) $$R_1(n,\ k,\ \lambda+m) = \sum_{j=0}^{n}\binom{n}{j}(m)_j R_1(n-j,\ k,\ \lambda).$$

We may write (5.13) in the form

(5.16) $$\Delta_\lambda R_1(n,\ k,\ \lambda) = nR_1(n-1,\ k,\ \lambda+1).$$

Iteration gives

(5.17) $$\Delta_\lambda^m R_1(n,\ k,\ \lambda) = n(n-1)\cdots(n-m+1)R_1(n-m,\ k,\ \lambda+m).$$

6. ORTHOGONALITY RELATIONS

Comparing (5.8) with (3.14), we have immediately the orthogonality relations

(6.1) $$\sum_{k=0}^{n}(-1)^{n-k}R(n,\ k,\ \lambda)R_1(k,\ j,\ \lambda) = \sum_{k=0}^{n}R_1(n,\ k,\ \lambda)\cdot(-1)^{k-j}R(k,\ j,\ \lambda) = \delta_{n,j},$$

the Kronecker delta.

It is of some interest to give a proof of (6.1) making use of (3.2) and (5.2). We have

$$\sum_{k=0}^{n}(-1)^{n-k}R(n,\ k,\ \lambda)R_1(k,\ j,\ \lambda)$$

$$= \sum_{k=0}^{n}(-1)^{n-k}\sum_{m=0}^{n-k}\binom{n}{m}\lambda^m S(n-m,\ k)\sum_{t=0}^{k-j}\binom{j+t}{t}\lambda^t S_1(k,\ k+t)$$

$$= \sum_{m=0}^{n}\sum_{t=0}^{n-j}(-1)^m\binom{n}{m}\binom{j+t}{t}\lambda^{m+t}\sum_{k=0}^{n-m}(-1)^{n-m-k}S(n-m,\ k)S_1(k,\ j+t).$$

The inner sum is equal to 1 if $n - m = j + t$, and vanishes otherwise. Thus, we have

$$\lambda^{n-j}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\binom{n-m}{j} = \lambda^{n-j}\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}\binom{m}{j} = \delta_{n,j},$$

so that

$$\sum_{k=0}^{n}(-1)^{n-k}R(n, k, \lambda)R_1(k, j, \lambda) = \delta_{n,j}. \tag{6.2}$$

As for the second half of (6.1), we have

$$\begin{aligned}
&\sum_{k=0}^{n}R_1(n, k, \lambda)\cdot(-1)^{k-j}R(k, j, \lambda)\\
&= \sum_{k=0}^{n}\sum_{t=0}^{n-k}\binom{t+k}{t}\lambda^t S_1(n, t+k)\cdot(-1)^{k-j}\sum_{m=0}^{k-j}\binom{k}{m}\lambda^m S(k-m, j)\\
&= \sum_{k=0}^{n}\sum_{t=k}^{n}\binom{t}{k}\lambda^{t-k}S_1(n, t)\cdot(-1)^{k-j}\sum_{m=j}^{k}\binom{k}{m}\lambda^{k-m}S(m, j)\\
&= \sum_{t=0}^{n}\sum_{m=j}^{n}(-1)^{t-j}\lambda^{t-m}S_1(n, t)S(m, j)\sum_{k=0}^{t}(-1)^{t-k}\binom{t}{k}\binom{k}{m}\\
&= \sum_{t=0}^{n}\sum_{m=j}^{n}(-1)^{t-j}\lambda^{t-m}S_1(n, t)S(m, j)\delta_{t,m}\\
&= \sum_{t=j}^{n}(-1)^{t-j}S_1(n, t)S(t, j) = \delta_{n,j}.
\end{aligned}$$

This, together with (6.2), completes the proof of (6.1).

The proof of (6.2) above suggests a more general result. As in the above proof, we have

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{n-k}R(n, k, \lambda)R_1(k, j, \mu) &= \sum_{m=0}^{n}\sum_{t=0}^{n-j}(-1)^m\binom{n}{m}\binom{j+t}{j}\lambda^m\mu^t\delta_{n-m,j+t}\\
&= \sum_{m=0}^{n}(-1)^m\binom{n}{m}\binom{n-m}{j}\lambda^m\mu^{n-m-j}\\
&= \sum_{m=j}^{n}(-1)^{n-m}\binom{n}{m}\binom{m}{j}\lambda^{n-m}\mu^{m-j}\\
&= \binom{n}{j}\sum_{m=1}^{n}(-1)^{n-m}\binom{n-j}{m-j}\lambda^{n-m}\mu^{m-j}\\
&= (-1)^{n-j}\binom{n}{j}\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}\lambda^{n-j-m}\mu^m,
\end{aligned}$$

and therefore,

$$(6.3)\qquad \sum_{k=0}^{n}(-1)^{n-k}R(n,\ k,\ \lambda)R_1(k,\ j,\ \mu) = \binom{n}{j}(\mu-\lambda)^{n-j}.$$

For $\mu = \lambda$, (6.3) reduces to (6.2).

In the next place

$$\sum_{k=0}^{n} R_1(n,\ k,\ \mu)\cdot(-1)^{k-j}R(k,\ j,\ \lambda)$$

$$= \sum_{k=0}^{n}\sum_{t=k}^{n}\binom{t}{k}\mu^{t-k}S_1(n,\ t)\cdot(-1)^{k-j}\sum_{m=j}^{k}\binom{k}{m}\lambda^{k-m}S(m,\ j)$$

$$= \sum_{t=0}^{n}\sum_{m=j}^{n}(-1)^{t-j}\binom{t}{m}S_1(n,\ t)S(m,\ j)\sum_{k=m}^{t}(-1)^{t-k}\binom{t-m}{k-m}\mu^{t-k}\lambda^{k-m}$$

$$= \sum_{t=0}^{n}\sum_{m=j}^{t}(-1)^{t-j}\binom{t}{m}S_1(n,\ t)S(m,\ j)(\lambda-\mu)^{t-m}.$$

Let $U(n,\ j)$ denote this sum. Then,

$$\sum_{j=0}^{n}(-1)^j U(n,\ j)j!\binom{x}{j} = \sum_{t=0}\sum_{m=0}(-1)^t\binom{t}{m}S_1(n,\ t)(\lambda-\mu)^{t-m}\sum_{j=0}^{m}S(m,\ j)j!\binom{x}{j}$$

$$= \sum_{t=0}^{n}\sum_{m=0}^{t}(-1)^t\binom{t}{m}S_1(n,\ t)(\lambda-\mu)^{t-m}x^m$$

$$= \sum_{t=0}^{n}(-1)^t S_1(n,\ t)(x+\lambda-\mu)^t$$

$$= (-1)^n(x+\lambda-\mu)(x+\lambda-\mu-1)\cdots(x+\lambda-\mu-n+1).$$

Replacing x by $-x$, this becomes

$$(6.4)\qquad \sum_{j=0}^{n}U(n,\ j)(x)_j = (x-\lambda+\mu)_n.$$

Since

$$(x+y)_n = \sum_{j=0}^{n}\binom{n}{j}(x)_j(y)_{n-j},$$

it follows from (6.4) that

$$U(n,\ j) = \binom{n}{j}(\mu-\lambda)_{n-j}.$$

Therefore, we have

$$(6.5)\qquad \sum_{k=0}^{n}R_1(n,\ k,\ \mu)\cdot(-1)^{k-j}R(k,\ j,\ \lambda) = \binom{n}{j}(\mu-\lambda)_{n-j}.$$

This result may be compared with (6.3). If we define matrices

$$M = [(-1)^{n-k}R(n, k, \lambda)] \qquad (n, k = 0, 1, 2, \ldots),$$

and

$$M_1 = [R_1(n, k, \mu)] \qquad (n, k = 0, 1, 2, \ldots),$$

then (6.3) and (6.5) become

(6.3)′
$$MM_1 = \left[\binom{n}{k}(\lambda - \mu)^{n-k}\right],$$

and

(6.5)′
$$M_1M = \left[\binom{n}{k}(\mu - \lambda)_{n-k}\right],$$

respectively.

7. COMBINATORIAL INTERPRETATION OF $R(n, k, \lambda)$ AND $R_1(n, k, \lambda)$

Let λ be a nonnegative integer and let $B_1, B_2, \ldots, B_\lambda$ denote λ open boxes. Let $P(n, k, \lambda)$ denote the number of partitions of $Z_n = \{1, 2, \ldots, n\}$ into k blocks with the understanding that an arbitrary number of the elements of Z_n may be placed in any number (possibly none) of the boxes. For brevity, we shall call these "λ-partitions." Clearly,

(7.1)
$$P(n, k, 0) = S(n, k).$$

To evaluate $P(n, 0, \lambda)$, we place x_1 elements of Z_n in B_1, x_2 in B_2, ..., x_λ in B_λ. Thus,

$$P(n, 0, \lambda) = \sum_{x_1 + x_2 + \cdots + x_\lambda} \frac{n!}{x_1!x_2! \ldots x_\lambda!}.$$

Hence,

(7.2)
$$P(n, 0, \lambda) = \lambda^n.$$

Also, clearly,

(7.3)
$$P(0, k, \lambda) = \delta_{0,k}.$$

To get a recurrence for $P(n, k, \lambda)$, we consider the effect of adding the element $n+1$ to a λ-partition of Z_n into k blocks. The added element may be placed in any of the blocks or any of the boxes without changing the value of k. On the other hand, if it constitutes an additional block, then of course the number of blocks becomes $k+1$. Thus, we have

(7.4)
$$P(n + 1, k, \lambda) = (\lambda + k)P(n, k, \lambda) + P(n, k - 1, \lambda).$$

Since

$$P(0, k, \lambda) = R(0, k, \lambda) = \delta_{0,k},$$

comparison of (7.4) with (3.6) gives

(7.5)
$$P(n, k, \lambda) = R(n, k, \lambda).$$

Hence, $R(n, k, \lambda)$ *is equal to the number of* λ*-partitions of* Z_n *into* k *blocks.*

Turning next to $R(n, k, \lambda)$, again let $B_1, B_2, \ldots, B_\lambda$ denote λ open boxes. Let $P_1(n, k, \lambda)$ denote the number of permutations of Z_n with k cycles with the understanding that an arbitrary number of the elements of Z_n may be placed in any number (possibly none) of the boxes and then permuted in all possible ways in each box. For brevity, we call these "λ-permutations."

Clearly,

(7.6) $$P_1(n, k, 0) = S_1(n, k).$$

To evaluate $P(n, 0, \lambda)$, note that $P(1, 0, \lambda) = \lambda$ and

$$P(n + 1, 0, \lambda) = (\lambda + n)P(n, 0, \lambda),$$

since the element $n + 1$ may occupy any one of the $n + \lambda$ positions. Thus,

(7.7) $$P_1(n, 0, \lambda) = (\lambda)_n.$$

Also clearly,

(7.8) $$P_1(0, k, \lambda) = \delta_{0,k}.$$

A recurrence for $P_1(n, k, \lambda)$ is obtained using the method of proof of (7.4); however, there are now $\lambda + n$ possible positions for the element $n + 1$. Thus, we get

(7.9) $$P_1(n + 1, k, \lambda) = (\lambda + n)P_1(n, k, \lambda) + P_1(n, k - 1, \lambda).$$

Comparison of (7.9) with (5.5) gives

(7.10) $$P_1(n, k, \lambda) = R_1(n, k, \lambda).$$

Hence, $R_1(n, k, \lambda)$ *is equal to the number of* λ*-permutations of* Z_n *with* k *cycles.*

We remark that (7.5) can also be proved using (3.2) and that (7.10) can be proved using (5.3).

Finally, we note that the generalized Bell number defined by (3.17),

$$B(n, \lambda) = \sum_{k=0}^{n} R(n, k, \lambda),$$

is equal to the total number of λ*-partitions of* Z_n.

REFERENCES

1. L. Comtet. *Advanced Combinatorics*. Boston: D. Reidel, 1974.
2. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: Wiley & Sons, 1958.

*****