$$\ell_n(r, 0, s) = \frac{t_1^n + t_2^n - t_3^n - t_4^n}{t_1 + t_2 - t_3 - t_4}.$$

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GEOMETRIC RECURRENCE RELATION

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1. INTRODUCTION

In a previous paper [1], we considered r, s sequences $\{U_k\}$ and obtained explicit formulations for the general term in powers of r and s. We noted 2 special sequences $\{G_k\}$ and $\{M_k\}$. These are sequences that specialize to the Fibonacci and Lucas sequences where $r = s = 1$.

In this paper, we propose to consider the relationship between r, s recurrence relations and geometric sequences. We give a necessary and sufficient condition on r and s for the recurrence relation to be geometric. We conclude the section by showing how to write any geometric sequence as an r, s recurrence relation.

In the final section, we briefly consider a special Fibonacci sequence. We give an explicit formulation for its general term. We are then able to note when it is a geometric sequence.

2. GEOMETRIC r, s SEQUENCES

In the previous paper [1] we considered the special r, s relations $\{G_k\}$ and $\{M_k\}$ which were characterized by the initial values $G_0 = 0$, $G_1 = 1$, $M_0 = 2$, and $M_1 = r$. We further specialize r and s so that the characteristic equation of the sequence has a multiple root λ. We then have $r = 2\lambda$ and $s = -\lambda^2$. It can be readily verified that the expression for the general terms are

$$G_k = k\lambda^{k-1} \quad \text{and} \quad M_k = 2\lambda^k.$$

Note that the M_k sequence is geometric with ratio of λ and first term of $M_0 = 2$. But the other sequence is not geometric. We shall develop the general conditions for which these two results are special cases.

Before going to the main theorem, we will make a few observations. Consider the general term of the r, s sequence $\{U_k\}$:

$$U_n = rU_{n-1} + sU_{n-2};\ U_0,\ U_1 \text{ arbitrary.}$$

If $s = 0$, this would be a geometric sequence starting with U_1. Further, if the initial values were such that $U_1 = rU_0$, the sequence would be geometric with U_0 as the first term.

If $r = 0$, we have two geometric sequences with ratio s. One of these is the even indexed U_k with U_0 as initial value. The other geometric sequence is the odd indexed U_k with U_1 as starting value.

We shall call these two cases the trivial cases. In other words, an r, s relation for which $rs = 0$ is trivially geometric.

There is a whole class of r, s sequences that are geometric only in this trivial case. These are the sequences, for which $U_0 = 0$, for in this case

$$U_2 = rU_1 + sU_0 = rU_1,$$
$$U_3 = rU_2 + sU_1 = (r^2 + s)U_1.$$

Now this is geometric only if $r^2 + s = r^2$. But this can only happen for $s = 0$. Included in this class is the $\{G_k\}$ sequence.

We shall assume in the rest of this section that U_0, r, and s are all nonzero. We are ready to state and prove our theorem.

Theorem 2.1: The r, s sequence $\{U_k\}$ is geometric if and only if

$$\frac{r + e}{2} = \frac{U_1}{U_0}, \text{ where } e = \pm\sqrt{r^2 + 4s}.$$

For convenience, we shall denote the ratio as m so that $r + e = 2m$ or $r = 2m - e$. We find that

$$s = \frac{e^2 - r^2}{4} = \frac{e^2 - (2m - e)^2}{4} = m(e - m).$$

We also need the result that

$$rm + s = 2m^2 - me + me - m^2 = m^2.$$

From the expression for U_2 and the assumption that $U_1 = mU_0$, we have

$$U_2 = rU_1 + sU_0 = r(mU_0) + sU_0 = (rm + s)U_0 = m^2U_0 = mU_1.$$

Assume that $U_k = mU_{k-1}$ for $k = 2, \ldots, i - 1$. For

$$U_i = rU_{i-1} + sU_{i-2} = r(mU_{i-2}) + sU_{i-2} = (rm + s)U_{i-2} = m^2U_{i-2} = mU_{i-1}.$$

Hence, the sequence is geometric with U_0 as first term and ratio of m.

Conversely, assume $\{U_k\}$ is geometric with ratio m so that $U_k = mU_{k-1}$ for all k. Since

$$U_k = rU_{k-1} + sU_{k-2} = (rm + s)U_{k-2},$$

and, by assumption,

$$U_k = mU_{k-1} = m(mU_{k-2}) = m^2U_{k-2},$$

it follows that $rm + s = m^2$. This means that m is a solution of the equation $x^2 - rx - s = 0$. The roots of this equation are $\frac{r \pm e}{2}$, so $m = \frac{r + e}{2}$. Further, $U_1 = mU_0$ so $\frac{U_1}{U_0} = m$. But these are the given equivalent conditions.

In the proof, it was not necessary that r and s be integers. The results are then valid for a more general recurrence relation. In the corollary that follows, we note how any geometric sequence can be expressed as an r, s relation.

Corollary 2.1: The geometric sequence $U_k = at^k$ can be represented as the r, s sequence with $U_0 = a$, $U_1 = at$, $r = 2t - \lambda$, $s = t\lambda - t^2$ for any λ.

By the choice of U_0 and U_1, we have $U_1 = tU_0$. Also,

$$e^2 = r^2 + 4s = 4t^2 - 4t\lambda + \lambda^2 + 4t\lambda - 4t^2 = \lambda^2,$$

so that

$$\frac{r + e}{2} = \frac{2t - \lambda + \lambda}{2} = t.$$

Hence, by the theorem, this r, s sequence is geometric.

3. A SPECIAL TRIBONACCI SEQUENCE

There is a special Tribonacci sequence that is geometric under some conditions. It can be verified that the sequence

$$T_n = rT_{n-1} + sT_{n-2} - rsT_{n-3};\ T_0,\ T_1,\ T_2 \text{ arbitrary}$$

has for a solution

$$T_{2k+2} = \sum_{j=0}^{k} r^{2k-2j} s^j (T_2 - sT_0) + s^{k+1}T_0;$$

$$T_{2k+3} = \sum_{j=0}^{k} r^{2k+1-2j} s^j (T_2 - sT_0) + s^{k+1}T_1.$$

The roots of the characteristic equation of the sequence are r, $\pm\sqrt{s}$. In case $T_2 - sT_0 = 0$, we see that the even-indexed terms form a geometric sequence with ratio s and initial value T_0. Note that the condition imposed has $T_2 = sT_0$. The odd-indexed terms also form a geometric sequence with ratio s and initial value T_1.

We have another important special case to be noted. If $T_0 = T_1 = 0$, we do not need to differentiate between even- and odd-indexed terms. We have for solution

$$T_m = \sum_{j=0}^{\left[\frac{m-2}{2}\right]} r^{m-2-2j} s^j T_2$$

if $T_2 = 1$, we have represented the restricted partitions of $m - 2$ as a sum of $(m - 2 - 2j)$ 1's and (j) 2's.

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REPRESENTATIONS FOR r, s RECURRENCE RELATIONS

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1. STATEMENT OF THE PROBLEM

Recently, Buschman [1], Horadam [2], and Waddill [3] considered properties of the recurrence relation

$$U_k = rU_{k-1} + sU_{k-2}$$

where r, s are nonnegative integers. Buschman and Horadam gave representations for U_k in powers of r and $e = (r^2 + 4s)^{1/2}$. In this paper we give them in powers of r and s. We write the K_n of Waddill as G_k. It is a generalization of the Fibonacci sequence. We also consider a sequence $\{M_k\}$ that is a generalization of the Lucas sequence.

For the $\{G_k\}$ and $\{M_k\}$ sequences, we obtain two representations for their general terms. From this, we move to a representation for the general term of the basic sequence. A computer program has been written that gives this term for specified values of the parameters.

In this paper we use some standard notation. We start by defining

$$e^2 = r^2 + 4s,$$

where e could be irrational. We also need to define

$$\alpha = (r + e)/2 \quad \text{and} \quad \beta = (r - e)/2.$$

In other words, α and β are solutions of the quadratic equation

$$x^2 - rx - s = 0.$$

We can easily show that $\alpha + \beta = r$, $\alpha - \beta = e$, and $\alpha\beta = -s$.

2. GENERALIZATIONS OF THE FIBONACCI AND LUCAS SEQUENCES

Using the α and β given in the first section, we can define two special r, s sequences. These are given by

$$G_k = \frac{\alpha^k - \beta^k}{e}(e \neq 0), \qquad M_k = \alpha^k + \beta^k.$$

It is easy to verify that

$$G_0 = 0,\ G_1 = 1,\ G_2 = r,\ G_3 = r^2 + s,\ G_4 = r^3 + 2rs;$$
$$M_0 = 2,\ M_1 = r,\ M_2 = r^2 + 2s,\ M_3 = r^3 + 3rs,$$
$$M_4 = r^4 + 4r^2s + 2s^2;$$

and that they satisfy the basic r, s recurrence relation; i.e.,