REFERENCE

1. L. E. Fuller. "Representations for r, s Recurrence Relations." Below.

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REPRESENTATIONS FOR r, s RECURRENCE RELATIONS

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1. STATEMENT OF THE PROBLEM

Recently, Buschman [1], Horadam [2], and Waddill [3] considered properties of the recurrence relation

$$U_k = rU_{k-1} + sU_{k-2}$$

where r, s are nonnegative integers. Buschman and Horadam gave representations for U_k in powers of r and $e = (r^2 + 4s)^{1/2}$. In this paper we give them in powers of r and s. We write the K_n of Waddill as G_k. It is a generalization of the Fibonacci sequence. We also consider a sequence $\{M_k\}$ that is a generalization of the Lucas sequence.

For the $\{G_k\}$ and $\{M_k\}$ sequences, we obtain two representations for their general terms. From this, we move to a representation for the general term of the basic sequence. A computer program has been written that gives this term for specified values of the parameters.

In this paper we use some standard notation. We start by defining

$$e^2 = r^2 + 4s,$$

where e could be irrational. We also need to define

$$\alpha = (r + e)/2 \quad \text{and} \quad \beta = (r - e)/2.$$

In other words, α and β are solutions of the quadratic equation

$$x^2 - rx - s = 0.$$

We can easily show that $\alpha + \beta = r$, $\alpha - \beta = e$, and $\alpha\beta = -s$.

2. GENERALIZATIONS OF THE FIBONACCI AND LUCAS SEQUENCES

Using the α and β given in the first section, we can define two special r, s sequences. These are given by

$$G_k = \frac{\alpha^k - \beta^k}{e}(e \neq 0), \quad M_k = \alpha^k + \beta^k.$$

It is easy to verify that

$$G_0 = 0,\ G_1 = 1,\ G_2 = r,\ G_3 = r^2 + s,\ G_4 = r^3 + 2rs;$$
$$M_0 = 2,\ M_1 = r,\ M_2 = r^2 + 2s,\ M_3 = r^3 + 3rs,$$
$$M_4 = r^4 + 4r^2s + 2s^2;$$

and that they satisfy the basic r, s recurrence relation; i.e.,

$$G_2 = rG_1 + sG_0 \qquad M_2 = rM_1 + sM_0$$
$$G_3 = rG_2 + sG_1 \qquad M_3 = rM_2 + sM_1$$
$$G_4 = rG_3 + sG_2 \qquad M_4 = rM_3 + sM_2$$

In the next theorem, we prove that these two sequences are indeed r, s sequences.

Theorem 1: The sequences $\{G_k\}$ and $\{M_k\}$ are r, s sequences.

The proofs for both utilize mathematical induction. We have already indicated the validity of the theorem for $k = 2$, 3, and 4. We assume the terms satisfy the r,s relation for $k = 2, 3, \ldots, i - 1$. We form

$$\begin{aligned} rG_{i-1} + sG_{i-2} &= (\alpha + \beta)\frac{\alpha^{i-1} - \beta^{i-1}}{e} + (-\alpha\beta)\frac{\alpha^{i-2} - \beta^{i-2}}{e} \\ &= \frac{\alpha^i - \beta^i + \alpha^{i-1}\beta - \alpha\beta^{i-1} - \alpha^{i-1}\beta + \alpha\beta^{i-1}}{e} \\ &= \frac{\alpha^i - \beta^i}{e}. \end{aligned}$$

This is G_i by definition, so this sequence is an r, s sequence.

For the second part, we once more assume that the terms satisfy the r, s relation for $k = 2, \ldots, i - 1$. We form this time

$$\begin{aligned} rM_{i-1} + sM_{i-2} &= (\alpha + \beta)(\alpha^{i-1} + \beta^{i-1}) + (-\alpha\beta)(\alpha^{i-2} + \beta^{i-2}) \\ &= \alpha^i + \beta^i + \alpha^{i-1}\beta + \alpha\beta^{i-1} - \alpha^{i-1}\beta - \alpha\beta^{i-1} \\ &= \alpha^i + \beta^i. \end{aligned}$$

This is M by definition, so this too is an r, s sequence.

We obtain the Fibonacci and Lucas sequences from these two by letting $r = s = 1$. This can be readily verified.

In the next two theorems we give a more explicit formulation for G_k and M_k that can be easily programmed for a computer.

Theorem 2: For the sequence $\{G_k\}$,

$$G_k = \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \binom{k-1-j}{j} r^{k-1-2j} s^j, \; k > 0; \; G_0 = 0.$$

We shall prove this by induction. We first note that this formulation for $k = 1, 2, 3, 4$ gives the same results as the previous one.

$$\begin{aligned} G_1 &= \binom{0}{0} r^0 s^0 = 1 \\ G_2 &= \binom{1}{0} r = r \\ G_3 &= \binom{2}{0} r^2 + \binom{1}{1} s = r^2 + s \\ G_4 &= \binom{3}{0} r^3 + \binom{2}{1} rs = r^3 + 2rs \end{aligned}$$

We assume that the result is valid for $k = 1, \ldots, i - 1$. We now show that $rG_{i-1} + sG_{i-2}$ does give the expression for G_i. Consider then

$$rG_{i-1} + sG_{i-2} = r\sum_{j=0}^{\left[\frac{i-2}{2}\right]} \binom{i-2-j}{j} r^{i-2-2j}s^j + s\sum_{j=0}^{\left[\frac{i-3}{2}\right]} \binom{i-3-j}{j} r^{i-3-2j}s^j$$

$$= \sum_{j=0}^{\left[\frac{i-2}{2}\right]} \binom{i-2-j}{j} r^{i-1-2j}s^j + \sum_{j=0}^{\left[\frac{i-3}{2}\right]} \binom{i-3-j}{j} r^{i-3-2j}s^{j+1}.$$

We now introduce a standard change that we use in several proofs. We first remove the first term of the first summation; then we shift the index of the second summation by replacing j by $j - 1$. This gives the same exponents for r and s in both summations. We then have

$$r^{i-1} + \sum_{j=1}^{\left[\frac{i-2}{2}\right]} \binom{i-2-j}{j} r^{i-1-2j}s^j + \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \binom{i-2-j}{j-1} r^{i-1-2j}s^j.$$

If i is even, the upper limits of both summations are equal, so we can combine them into the single summation:

$$r^{i-1} + \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \left[\binom{i-2-j}{j} + \binom{i-2-j}{j-1}\right] r^{i-1-2j}s^j$$

$$= r^{i-1} + \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \binom{i-1-j}{j} r^{i-1-2j}s^j.$$

We see that the summand is r^{i-1} for $j = 0$. We include that term in the summation and obtain the desired expression for G_i.

If i is odd, then the upper limit on the second summation is one larger than that on the first. We break off the last term on the second summation and combine the two summands. This gives

$$r^{i-1} + \sum^{\left[\frac{i-3}{2}\right]} \left[\binom{i-2-j}{j} + \binom{i-2-j}{i-1}\right] r^{i-1-2j}s^j + s^{(i-1)/2}$$

$$= r^{i-1} + \sum^{\left[\frac{i-3}{2}\right]} \binom{i-1-j}{j} r^{i-1-2j}s^j + s^{(i-1)/2}.$$

We see that the summand gives r^{i-1} for $i = 0$ and $s^{(i-1)/2}$ for $i = \left[\frac{i-1}{2}\right]$. We combine these terms into the summation and we have the expression for G_i.

Hence, in any case, we do obtain the desired formula for G_i, so it must be valid for all terms of the sequence.

In passing, we might note that for the Fibonacci sequence we have

$$F_k = \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \binom{k-1-j}{j},\ k > 0;\ F_0 = 0.$$

In the next theorem for the $\{M_k\}$, we need the following property of binomial coefficients:

$$\frac{i-1}{i-1-j}\binom{i-1-j}{j} + \frac{i-2}{i-1-j}\binom{i-1-j}{j-1} = \frac{i}{i-j}\binom{i-j}{j}.$$

This can be readily verified using factorials.

<u>*Theorem 3:*</u> For the sequence $\{M_k\}$,

$$M_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{k}{k-j}\binom{k-j}{j} r^{k-2j}s^j, \quad k > 0; \; M_0 = 2.$$

The proof is by induction, so we first note that it is valid for $k = 1, 2, 3$.

$$M_1 = \sum_{j=0}^{0} \frac{1}{1-j}\binom{1-j}{j} r^{1-2j}s^j = \frac{1}{1}\binom{1}{0} r^1 s^0 = r;$$

$$M_2 = \sum_{j=0}^{1} \frac{2}{2-j}\binom{2-j}{j} r^{2-2j}s^j = \frac{2}{2}\binom{2}{0} r^2 + \frac{2}{1}\binom{1}{1} s = r^2 + 2s;$$

$$M_3 = \sum_{j=0}^{1} \frac{3}{3-j}\binom{3-j}{j} r^{3-2j}s^j = \frac{3}{3}\binom{3}{0} r^3 + \frac{3}{2}\binom{2}{1} r^2 = r^3 + 3r^2.$$

We assume that the formula is valid for $k = 2, 3, \ldots, i-1$ and show it is valid for M . The proof is similar to that of Theorem 2 except that we have an extra term for the case i is even.

We start with the basic

$$rM_{i-1} + sM_{i-2} = \sum_{j=0}^{\left[\frac{i-1}{2}\right]} \frac{i-1}{i-1-j}\binom{i-1-j}{j} r^{i-2j}s^j + \sum_{j=0}^{\left[\frac{i-2}{2}\right]} \frac{i-2}{i-2-j}\binom{i-2-j}{j-1} r^{i-2-2j}s^{j+1}.$$

Once more we break off the first term in the first summation and shift the second summation index to give

$$r^i + \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \frac{i-1}{i-1-j}\binom{i-1-j}{j} r^{i-2j}s^j + \sum_{j=1}^{\left[\frac{i}{2}\right]} \frac{i-2}{i-1-j}\binom{i-1-j}{j-1} r^{i-2j}s^j.$$

If i is odd, the two summations have the same upper limit; thus, we can combine them using the property of binomial coefficients given before the theorem. This gives, for the summation,

$$r^i + \sum_{j=1}^{\left[\frac{i}{2}\right]} \frac{i}{i-j}\binom{i-j}{j} r^{i-2j}s^j.$$

Finally, note that the summand is r^i for $j = 0$. We combine into a single sum that is the formula for M_i.

In case i is even, the second summation has an extra term of $2s^{i/2}$. If we separate it from the summation, we can combine the two summations to get

$$r^i + \sum_{j=1}^{\left[\frac{i-2}{2}\right]} \frac{i}{i-j}\binom{i-j}{j} r^{i-2j}s^j + 2s^{i/2}.$$

The summand is r^i for $j = 0$ and $2s^{i/2}$ for $j = i/2$, so we can combine these and obtain the expression for M_i. Hence, in either case, the formula is valid for all integers k.

This theorem gives, for the general term of the Lucas sequence,

$$L_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{k}{k-j}\binom{k-j}{j}, \; k > 0; \; L_0 = 2.$$

3. THE FORMULATION FOR U_k

In this section, we first prove a basic result for $\{U_k\}$. It is comparable to the result in Waddill's paper for $K_n = G_n$.

Theorem 4: The general term of $\{U_k\}$ can be expressed as

$$U_k = U_{t+j} = G_j U_{t+1} + G_{j-1} s U_t.$$

Once more the proof is by induction. For $j = 2$, we have

$$U_{t+2} = G_2 U_{t+1} + G_1 s U_t = r U_{t+1} + s U_t,$$

which is true for all t. Assume that the expression is true for $j = 2, \ldots, i - 1$. Then, since U_{t+i} is an r, s sequence,

$$\begin{aligned} U_{t+i} &= rU_{t+i-1} + sU_{t+i-2} = r(G_{i-1}U_{t+1} + G_{i-2}sU_t) + s(G_{i-2}U_{t+1} + G_{i-3}sU_t) \\ &= (rG_{i-1} + sG_{i-2})U_{t+1} + (rG_{i-2} + sG_{i-3})sU_t = G_i U_{t+1} + G_{i-1}U_t. \end{aligned}$$

Hence, the result is true for $j = i$ and so is true for all integers.

We can now give a formulation for U_k in terms of its initial values U_0 and U_1. This is given in the next theorem.

Theorem 5: The general term of the r, s sequence $\{U_k\}$ is given by

$$U_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j.$$

In Theorem 4, we take $t = 0$, so $j = k$, and we have

$$U_k = G_k U_1 + G_{k-1} s U_0.$$

Substituting the result of Theorem 2 for G_k, G_{k-1},

$$U_k = \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \binom{k-1-j}{j} r^{k-1-2j}s^j U_1 + \sum_{j=0}^{\left[\frac{k-2}{2}\right]} \binom{k-2-j}{j} r^{k-2-2j}s^j (sU_0).$$

Once more we break off the first term of the first summation and shift the index of the second summation to give

$$r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k-1}{2}\right]} \binom{k-1-j}{j} r^{k-1-2j}s^j U_1 + \sum_{j=1}^{\left[\frac{1}{2}\right]} \binom{k-1-j}{j-1} r^{k-2j}s^j U_0.$$

Again, we consider the two cases where k is odd or even. For k odd, the two upper indices are equal, so we can combine the two summations to obtain

$$r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k}{2}\right]} \left(\binom{k-1-j}{j} U_1 + \binom{k-1-j}{j-1} rU_0\right] r^{k-1-2j}s^j.$$

It can be verified that the summand can be written so that we have

$$\begin{aligned} U_k &= r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j \\ &= \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j \end{aligned}$$

For k even, we break off the last term in the second summation and have

$$\begin{aligned} & r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k-2}{2}\right]} \left(\binom{k-1-j}{j} U_1 + \binom{k-1-j}{j-1} rU_0\right] r^{k-1-2j}s^j + s^{k/2}U_0 \\ &= r^{k-1}U_1 + \sum_{j=1}^{\left[\frac{k-2}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j + s^{k/2}U_0 . \end{aligned}$$

we note that the summand gives $r^{k-1}U_1$ for $j = 0$ and $s^{k/2}U_0$ for $j = k/2$. Thus we can write, for the general k,

$$U_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j}s^j .$$

It can be verified that by letting $U_1 = M_1 = r$ and $U_0 = M_0 = 2$, we obtain the expression for M_k given in Theorem 3.

We can obtain an expression for $\{U_k\}$ in terms of $\{M_k\}$. This is shown in the next theorem.

Theorem 6: The $\{U_k\}$ is given by

$$U_k = U_{t+j} = \frac{M_1M_j + sM_0M_{j-1}}{M_1^2 + sM_0^2} U_{t+j} + \frac{M_1M_{j-1} + sM_0M_{j-2}}{M_1^2 + sM_0^2} U_t .$$

We can obtain this result from Theorem 4 by determining G_j and G_{j-1} in terms of $\{M_k\}$. For this, we start with

$$M_{j-1} = G_{j-1}M_1 + G_{j-2}sM_0 = rG_{j-1} + 2sG_{j-2}.$$

Since $G_j = rG_{j-1} + sG_{j-2}$, it follows that $2sG_{j-2} = 2G_j - 2rG_{j-1}$. We substitute this into the expression for M_{j-1}, and also write the expression for M_j to give the two equations:

$$M_{j-1} = 2G_j - rG_{j-1};$$
$$M_j = rG_j + 2sG_{j-1}.$$

The solutions for G_i and G_{i-1} are

$$G_j = \frac{rM_j + 2sM_{j-1}}{r^2 + 4s} = \frac{M_1M_j + sM_0M_{j-1}}{M_1^2 + sM_0^2}$$

and

$$G_{j-1} = \frac{2M_j - rM_{j-1}}{r^2 + 4s} = \frac{2(rM_{j-1} + sM_{j-2}) - rM_{j-1}}{r^2 + 4s} = \frac{M_1M_{j-1} + sM_0M_{j-2}}{M_1^2 + sM_0^2}.$$

Substituting the results in the expression for U_k of Theorem 4 gives the required expression for this theorem.

The formulation for U_k given in Theorem 5 has been programmed by Robert C. Fitzgerald. He is a senior in Computer Science. We can generate the U_k for specified values of r, s, U_1 and U_0.

Special cases of this result for $e = 0$ and other particular values of r and s will be considered in a future paper.

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THORO'S CONJECTURE AND ALLIED DIVISIBILITY PROPERTY OF LUCAS NUMBERS

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In [3], Thoro made a conjecture that for any prime $p \equiv 3 \pmod 4$, the congruence $F_{2n+1} \equiv 0 \pmod p$ is not solvable where F_{2n+1} is an arbitrary Fibonacci number of odd index. The conjecture has already been proved. In what follows, we give a different proof of this and discuss another problem that arose during this investigation.

Proof: If possible, let the above congruence be true: since $F_{2n+1} = F_n^2 + F_{n+1}^2$ (see [1], p. 56), we get

(1) $$F_n^2 + F_{n+1}^2 \equiv 0 \pmod p$$

Under this hypothesis, it follows that p divides neither F_n nor F_{n+1}. This