RECURSIVE, SPECTRAL, AND SELF-GENERATING SEQUENCES

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Let p be a fixed integer greater than 1 and define u_n for all integers n by

(1) $$u_0 = 0,\ u_1 = 1,\ u_{n+2} = pu_{n+1} + u_n.$$

Then $u_1, u_2, \ldots$ is an increasing sequence of integers with $u_1 = 1$ and hence a function $\sigma(n)$ is well defined for all n in $N = \{0, 1, 2, \ldots\}$ by

(2) $$\sigma(0) = 0,\ \sigma(n) = u_{j+1} + \sigma(n - u_j) \text{ for } u_j \le n < u_{j+1}.$$

Let $s = (p + \sqrt{p^2 + 4})/2$ and $S_n = [ns]$, where $[x]$ denotes the greatest integer in x.

It is shown below that the spectral sequence $\{S_n\}$ and the *shift function* $\sigma(n)$ are related by the equation

(3) $$S_n = u_2 + \sigma(n - 1)$$

and that $\{S_n\}$ has the self-generating property that

(4) $$S_{n+1} - S_n = \begin{cases} p \text{ if } n \text{ is not in } A = \{S_1, S_2, S_3, \ldots\}; \\ p + 1 \text{ if } n \text{ is in } A. \end{cases}$$

Also investigated are representations of positive integers in terms of $\{u_n\}$, partitions of $Z^+ = \{1, 2, \ldots\}$ into several sequences related to $\sigma(n)$ or S_n, the function counting the number of integers in $A \cap \{1, 2, \ldots, n\}$, and properties of "triangles" of entries $\begin{bmatrix} n \\ k \end{bmatrix}$ defined, for certain fixed x, by

$$\begin{bmatrix} n \\ k \end{bmatrix} = [nx] - [kx] - [(n - k)x] \text{ for } k = 0, 1, \ldots, n.$$

Most of the results presented here are analogous to those given in the authors' paper [4] in which the role of the present u_n is played by h_n satisfying

$$h_i = 2^{i-1} \text{ for } 1 \le i \le d,\ h_{n+d} + h_n = h_{n+1} + \cdots + h_{n+d-1}.$$

The Fibonacci numbers F_{n+1} are the case of the h_n with $d = 2$. The Fibonacci numbers could also be dealt with here by allowing p to equal 1; then the sequence $u_1, u_2, \ldots$ must be replaced by $u_2, u_3, \ldots$ in defining $\sigma(n)$.

For a bibliography on spectra of numbers, see [3].

1. PROPERTIES OF u_n

Here we state the properties of the u_n used below. Proofs are omitted since they are well known or easily derived, or both. Let $r_n = u_{n+1}/u_n$ for n in Z^+.

<u>*Lemma 1:*</u>

(a) For every k in Z^+, there is exactly one j in Z^+ with $u_j \le k < u_{j+1}$.
(b) $r_1 < r_3 < r_5 < \cdots < s < \cdots < r_6 < r_4 < r_2$.

(c) $u_{n+1}^2 - u_n u_{n+2} = (-1)^n$ for all n in Z.

(d) $r_n - r_{n+1} = (-1)^n/(u_n u_{n+1})$ for n in Z^+.

(e) gcd $(u_n, u_{n+1}) = 1$ for all n in Z.

(f) $u_{2n} = p(u_{2n-1} + u_{2n-3} + \cdots + u_1)$ for n in Z^+.

(g) $u_{2n-1} = p(u_{2n-2} + u_{2n-4} + \cdots + u_2) + u_1$ for n in Z^+.

2. RATIONAL APPROXIMATION

Let x be a positive irrational number. Then, we define a *Farey quadruple* for x to be an ordered quadruple (a, b, c, d) of positive integers, such that $bc - ad = 1$ and $a/b < x < c/d$.

The following result slightly extends some material from the theory of Farey sequences. (See [5] for background.)

Lemma 2: Let (a, b, c, d) be a Farey quadruple for x and let k be a positive integer less than $b + d$. Then:

(a) There is no integer h such that $a/b < h/k < c/d$.
(b) $[kx] = [ka/b]$.
(c) If $d \nmid k$, $[kx] = [kc/d]$.
(d) If $k = de$ with e in $\{1, 2, \ldots, b - 1\}$, $[kx] = [kc/d] - 1$.

The proofs are left to the reader.

We note that parts (b) and (c) of Lemma 1 tell us that

$$(u_{2m+2}, u_{2m+1}, u_{2m+1}, u_{2m}) \quad \text{and} \quad (u_{2m}, u_{2m-1}, u_{2m+1}, u_{2m})$$

are Farey quadruples for s whenever m is a positive integer. This is extended in the following result.

Lemma 3: Let $p \in \{2, 3, \ldots\}$, $s = (p + \sqrt{p^2 + 4})/2$, u be as in (1), and $m \in Z^+$. Then each of

$$(p, 1, 1 + kp, k) \text{ for } k = 1, 2, \ldots, p;$$

$$(u_{2m} + ku_{2m+1}, u_{2m-1} + ku_{2m}, u_{2m+1}, u_{2m}) \text{ for } k = 0, 1, \ldots, p;$$

$$(u_{2m+2}, u_{2m+1}, u_{2m+1} + ku_{2m+2}, u_{2m} + ku_{2m+1}) \text{ for } k = 0, 1, \ldots, p;$$

is a Farey quadruple for s.

Proof: Let (a, b, c, d) represent one of these quadruples. The property

$$bc - ad = 1$$

is easily verified using Lemma 1(c). The property

$$a/b < s < c/d$$

can be shown using Lemma 1(b) and the fact that

$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$$

whenever b and d are positive and $a/b < c/d$.

3. SPECTRA

Let $[x]$ denote the greatest integer in x, that is, the integer such that $[x] \leq x < [x] + 1$. The sequence $[x], [2x], [3x], \ldots$ is called the spectrum of x. It is a well-known result [1] that if y is an irrational number greater than 1 and $(1/x) + (1/y) = 1$ then the spectra $\{[nx]\}$ and $\{[ny]\}$ partition the positive integers Z^+.

Let p be in $\{2, 3, 4, \ldots\}$, $s = (p + \sqrt{p^2 + 4})/2$, $x = s - p + 1$, and $y = s + 1$. Also let $S_n = [ns]$, $X_n = [nx]$, and $Y_n = [ny]$. It is easily seen that y is irrational, $y > 1$, and $(1/x) + (1/y) = 1$; hence the spectra $\{X_n\}$ and $\{Y_n\}$ partition Z^+. It is also clear that $Y_n = X_n + np$ and that each of X_n and Y_n is an increasing function of n. It follows that $\{X_n\}$ and $\{Y_n\}$ may be self-generated using the following algorithm.

(5) $X_1 = 1$, $Y_1 = 1 + p$, X_k for $k > 1$ is the smallest positive integer not in the set $\{X_1, Y_1, X_2, Y_2, \ldots, X_{k-1}, Y_{k-1}\}$, and $Y_k = X_k + kp$.

Then $\{S_n\}$ is easily obtained from $S_n = Y_n - n = X_n + n(p - 1)$. It is shown below that $\{S_n\}$ can be self-generated from the initial condition $S_1 = p$ and the difference property (4) above.

The following result gives symmetry properties of finite segments $[x]$, ..., $[ex]$ of a spectrum for the cases in which e is the b or d of a Farey quadruple (a, b, c, d) for x.

Lemma 4: Let (a, b, c, d) be a Farey quadruple for x. Then:

(a) $[bx] = [kx] + [(b - k)x] + 1$ for $k = 1, 2, \ldots, b - 1$;
(b) $[dx] = [kx] + [(d - k)x]$ for $k = 0, 1, \ldots, d$.

Proof of (a): We have $[bx] = a$ from Lemma 2(b). Let $0 < k < b$, $j = b - k$, $h = [kx]$, and $i = [jx]$. Since x is irrational, $h < kx$ and so $h/k < x$. This, $x < c/d$, $k < b$, and Lemma 2(a) imply that $h/k < a/b$. Similarly, $i/j < a/b$. Since $(h + i)/(k + j)$ is in the closed interval with endpoints h/k and i/j, we have $(h + i)/(k + j) < a/b$. As $k + j = b$, this means that $h + i < a$ or $[kx] + [jx] < [bx]$. Then the desired result follows from the fact that, for all real y and z,

$$[y + z] - [y] - [z] \in \{0, 1\}. \tag{6}$$

Proof of (b): Lemma 2(d) tells us that $[dx] = c - 1$. We only need consider the k with $0 < k < d$. Let $j = d - k$, $[kx] = h$, and $[jx] = i$. Then $h + 1 > kx$ and so $(h + 1)/k > x$. This, $x > a/b$, $k < d$, and Lemma 2(a) then imply that $(h + 1)/k > c/d$. Similarly, $(i + 1)/j > c/d$, and hence $(h + 1 + i + 1)/(k + j) > c/d$. As $k + j = d$, one has $h + i + 2 > c$, which implies

$$[kx] + [(d - k)x] + 1 > [dx].$$

Again, the desired result follows from (6).

4. THE SHIFT PROPERTY

When convenient, $S_n = [ns]$ will also be denoted by $S(n)$. Also, we recall that $\sigma(n)$ is defined in (2) and u_j is defined in (1).

Theorem 1: If $u_j < n < u_j + u_{j+1}$ and $j \in Z^+$, then $S(n) = u_{j+1} + S(n - u_j)$.

Proof: Let (a, b, c, d) be the Farey quadruple $(u_{2m}, u_{2m-1}, u_{2m+1}, u_{2m})$ for s. Then Lemma 2(b) tells us that $S(n) = [ns] = [nr_{2m-1}]$ for $0 < n < u_{2m-1} + u_{2m}$. Hence

$$S(n) = [nu_{2m}/u_{2m-1}] = \left[\frac{u_{2m-1}u_{2m} + (n - u_{2m-1})u_{2m}}{u_{2m-1}}\right] = u_{2m} + S(n - u_{2m-1}) \tag{7}$$

for $u_{2m-1} < n < u_{2m-1} + u_{2m}$.

Next we use the Farey quadruple $(u_{2m+2}, u_{2m+1}, u_{2m+1}, u_{2m})$ for s and we find, from Lemma 2(c) and (d), that

$$S(n) = [nr_{2m}] \text{ if } 0 < n < u_{2m} + u_{2m+1} \text{ and } u_{2m} \nmid n,$$

$$S(n) = [nr_{2m}] - 1 \text{ if } n = ku_{2m} \text{ with } k \text{ in } \{1, 2, \ldots, u_{2m+1} - 1\}.$$

Using these facts, one can verify that

(8) $$S(n) = u_{2m+1} + S(n - u_{2m}) \text{ for } u_{2m} < n < u_{2m} + u_{2m+1}.$$

The desired result follows from (7) when j is odd and from (8) when j is even.

Theorem 2: $S_n = u_2 + \sigma(n - 1)$ for n in Z^+.

Proof: Since $S_1 = p = u_2$ and $\sigma(0) = 0$, the result holds for $n = 1$. Then a strong induction establishes it for all positive integers n using the consequence

$$S(n) = u_{j+1} + S(n - u_j) \text{ for } u_j < n \le u_{j+1}$$

of Theorem 1 and the consequence

$$\sigma(n - 1) = u_{j+1} + \sigma(n - 1 - u_j) \text{ for } u_j < n \le u_{j+1}$$

of the definition (2).

5. SEQUENCES OF COEFFICIENTS

Let V be the set of all sequences $E = [e_1, e_2, \ldots]$ with each e_i in $\{0, 1, \ldots, p\}$, with an i_0 such that $e_i = 0$ for $i > i_0$, and with $e_i = p$ implying that both $i > 1$ and $e_{i-1} = 0$. For such E, the sum

$$e_1u_{n+1} + e_2u_{n+2} + e_3u_{n+3} + \cdots$$

is actually a finite sum which we denote by $E \cdot U_n$. Also, we let $E \cdot U$ stand for $E \cdot U_0$.

Lemma 4: If E and E' are in V and $E \cdot U = E' \cdot U$, then $E = E'$.

This is shown using parts (f) and (g) of Lemma 1.

Theorem 3: The sequences of V form a sequence $E_0, E_1, E_2, \ldots$ such that

$$E_m \cdot U = m.$$

Proof: The only E in V with $E \cdot U = 0$ is $[0, 0, \ldots]$, which we denote by E_0. Now we assume that $k > 0$, and that there is a unique E_m in V with $E_m \cdot U = m$ for $m = 0, 1, \ldots, k - 1$. By Lemma 1(a), $u_j \le k < u_{j+1}$ for some j in Z^+. Let $h = k - u_j$; then we can let $[e_{h1}, e_{h2}, \ldots]$ be the unique E_h in V with $E_h \cdot U = h$. Then let $e_{kj} = 1 + e_{hj}$, $e_{ki} = e_{hi}$ for $i \ne j$, and $E_k = [e_{k1}, e_{k2}, \ldots]$. Since

$$k < u_{j+1} = pu_j + u_{j-1} < (p + 1)u_j,$$

one sees that $e_{kj} \le p$ and that if $e_{kj} = p$, then $j > 1$ and $e_{k,j-1} = 0$. Thus, E_k is in V. Clearly,

$$E_k \cdot U = E_h \cdot U + u_j = h + u_j = k.$$

Finally, there is no other E in V with $E \cdot U = k$ by Lemma 4.

The case with $p = 2$ of Theorem 3 was shown in [2].

6. PARTITIONING V

We now partition V into subsets V_1, V_2, V_3 and use these subsets to indicate the relationship of E_{m+1} to E_m. Let $E = [e_1, e_2, \ldots]$ be in V; then, E is in V_1 if $e_1 = p - 1$, E is in V_2 if $e_1 = 0$ and $e_2 = p$, and E is in V_3 if $e_1 < p - 1$ and $e_2 < p$. Since $e_1 > 0$ implies $e_2 < p$, one sees that each E of V is in one and only one of the V .

Lemma 5: Let $E_m = [e_1, e_2, \ldots]$ and $E_{m+1} = [f_1, f_2, \ldots]$. Then:

(a) If E_m is in V_1, let j be the smallest positive integer such that $e_{2j+1} < p$; then $f_i = 0$ for $i < 2j$, $f_{2j} = 1 + e_{2j}$, and $f_i = e_i$ for $i > 2j$.

(b) If E_m is in V_2, let h be the smallest positive integer such that $e_{2h} < p$; then $f_i = 0$ for $1 \leq i \leq 2h - 2$, $f_{2h-1} = 1 + e_{2h-1}$, and $f_i = e_i$ for $i \geq 2h$.

(c) If E_m is in V_3, $f_1 = 1 + e_1$ and $f_i = e_i$ for $i > 1$.

Proof: If we let $F = [f_1, f_2, \ldots]$ with the f_i as in (a), (b), and (c), it is easily seen that F is in V and $F \cdot U = 1 + E_m \cdot U = 1 + m$. This and Theorem 3 establish the present result.

Lemma 6: Let $\Delta_n(m) = E_{m+1} \cdot U_n - E_m \cdot U_n$. Then:

(a) $\Delta_n(m) = u_n + u_{n+1}$ if E_m is in V_1.

(b) $\Delta_n(m) = u_{n+1}$ if E_m is in V_2 or V_3.

Proof: These statements are easily verified using the parts of Lemma 5.

7. POWERS OF σ

Let $E_m = [e_{m1}, e_{m2}, \ldots]$ and let h be the largest i with $e_{mi} \neq 0$, then one can use the definition of σ in (2) to show that

$$\sigma(m) = \sigma(e_{m1}u_1 + \cdots + e_{mh}u_h) = e_{m1}u_2 + \cdots + e_{mh}u_{h+1} = E_m \cdot U_1.$$

Hence, there is no contradiction in defining σ^n for all integers n to be the function from N to Z given by

$$\sigma^n(m) = E_m \cdot U_n = e_{m1}u_{n+1} + e_{m2}u_{n+2} + \cdots . \tag{9}$$

Also let a_n be the function from Z^+ to Z defined by

$$a_n(k) = u_{n+1} + \sigma^n(k - 1). \tag{10}$$

We note that $a_0(k) = k$, that $a_1(k) = S_k$, and that, for fixed k, the $a_n(k)$ satisfy the same recurrence as the u_n, i.e.,

$$a_{n+2}(k) = pa_{n+1}(k) + a_n(k).$$

We also let A_n be the image set of a_n, i.e.,

$$A_n = \{a_n(k) : k \in Z^+\}.$$

Lemma 7: For n in $\{1, 2\}$, $A_n = \{i + 1 : E_i \in V_n\}$.

Proof: Using (10) and (9), one sees that

$$a_n(m + 1) = (1 + e_{m1})u_{n+1} + e_{m2}u_{n+2} + e_{m3}u_{n+3} + \cdots . \tag{11}$$

As m takes on all values in N, $F_m = [p - 1, e_{m1}, e_{m2}, \ldots]$ ranges through all the E_j in V_1 and $G_m = [0, p, e_{m1}, e_{m2}, \ldots]$ ranges through all the E_h in V_2. It follows from (11), Lemma 5, and the recursion in (1) that if $F_m = E_j$ then

$$j + 1 = E_{j+1} \cdot U = a_1(m + 1)$$

and, similarly, that if $G_m = E_h$ then

$$h + 1 = E_{h+1} \cdot U = a_2(m + 1).$$

These facts establish the lemma.

8. SELF-GENERATING SEQUENCES

Clearly, $a_n(1) = u_{n+1}$. This, and the following result, provide an easy self-generating rule for obtaining the sequence $\{a_1(k)\}$ and a similar easy rule for using $\{a_1(k)\}$ to obtain any $\{a_n(k)\}$.

Theorem 4: For n in Z and j in Z^+, $a_n(j + 1) - a_n(j)$ equals $u_n + u_{n+1}$ if j is in $A_1 = \{a_1(k) : k \in Z^+\}$ and equals u_{n+1} otherwise.

Proof: Lemma 7 tells us that $A_1 = \{j : E_{j-1} \in V_1\}$. Also,

$$a_n(j + 1) - a_n(j) = \sigma^n(j) - \sigma^n(j - 1) = E_j \cdot U_n - E_{j-1} \cdot U_n.$$

Hence, the desired result follows from Lemma 6.

Theorem 5: The number of integers in $A_1 \cap \{1, 2, \ldots, m\}$ is $a_{-1}(m + 1)$.

Proof: Let $\Delta_{-1}(i) = a_{-1}(i + 1) - a_{-1}(i)$. Clearly,

$$a_{-1}(m + 1) = a_{-1}(1) + \Delta_{-1}(1) + \Delta_{-1}(2) + \cdots + \Delta_{-1}(m). \tag{12}$$

Now $a_{-1}(1) = u_0 + \sigma^{-1}(0) = 0 + 0 = 0$. Also, Theorem 4 tells us that $\Delta_{-1}(i) = u_0 = 0$ when i is not in A_1 and $\Delta_{-1}(i) = u_0 + u_{-1} = 1$ when i is in A_1. Thus, the sum on the right side of (12) is the number of i that are in both $\{1, 2, \ldots, m\}$ and A_1, as desired.

9. PARTITIONING Z^+

We saw in Lemma 7 that $A_n = \{i + 1 : E_i \in V_n\}$ for n in $\{1, 2\}$. Let $B = \{j + 1 : E_j \in V_3\}$. Since V_1, V_2, V_3 is a partitioning of $V = \{E_0, E_1, \ldots\}$, it follows that A_1, A_2, B is a partitioning of $Z^+ = \{1, 2, \ldots\}$.

For $k = 1, 2, \ldots, p - 1$, we let

$$b_k(n) = a_1(n) + k - p = k + \sigma(n - 1)$$

and let

$$B_k = \{b_k(n) : n \in Z^+\}.$$

It is easily seen that

$$B_k = \{m : e_{m1} = k,\ e_{m2} < p\} \text{ for } 1 \le k < p$$

and that $B_1, B_2, \ldots, B_{p-1}$ is a partitioning of B. Hence, the sequences

$$\{b_1(n)\}, \{b_2(n)\}, \ldots, \{b_{p-1}(n)\}, \{a_1(n)\}, \{a_2(n)\}$$

partition the positive integers.

10. SPECTRUM TRIANGLES

Let x be irrational and greater than 1 and let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote $[nx] - [nk] - [(n - k)x]$ for integers n and k with $0 \le k \le n$. It now follows from (6) that

$\begin{bmatrix} n \\ k \end{bmatrix}$ is always in $\{0, 1\}$. The fact that $\begin{bmatrix} n \\ 0 \end{bmatrix} = 0 = \begin{bmatrix} n \\ n \end{bmatrix}$ and the symmetry property $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n - k \end{bmatrix}$ are obvious. Part (c) of the following result implies other symmetries for certain finite subtriangles of the infinite triangle of values of $\begin{bmatrix} n \\ k \end{bmatrix}$.

Theorem 6: Let (a, b, c, d) be a Farey quadruple for x. Then:

(a) $\begin{bmatrix} b \\ k \end{bmatrix} = 1$ for $0 < k < b$.

(b) $\begin{bmatrix} d \\ k \end{bmatrix} = 0$ for $0 \le k \le d$.

(c) $\begin{bmatrix} d - s + t \\ t \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$ for $0 \le t \le s \le d$.

Proof: Parts (a) and (b) are a restatement of Lemma 4. For (c) we use Lemma 4(b), or the present part (b), to see that

$$[dx] = [(s - t)x] + [(d - s + t)x] = [sx] + [(d - s)x].$$

Hence $[(d - s + t)x] - [(d - s)x] = [sx] - [(s - t)x]$, and so

$$\begin{aligned} \begin{bmatrix} d - s + t \\ t \end{bmatrix} &= [(d - s + t)x] - [tx] - [(d - s)x] \\ &= [sx] - [tx] - [(s - t)x] = \begin{bmatrix} s \\ t \end{bmatrix} \end{aligned}$$

as desired.

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