REFERENCES

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton-Mifflin Company, 1969.
2. V. E. Hoggatt, Jr., & G. E. Bergum. "Divisibility and Congruence Relations." *The Fibonacci Quarterly* 2 (1974):189.
3. D. Thoro. "Two Fibonacci Conjectures." *The Fibonacci Quarterly* 3 (1965): 184-186.

*****

A CLASS OF SOLUTIONS OF THE EQUATION $\sigma(n) = 2n + t$

NEVILLE ROBBINS
Bernard M. Baruch College, New York, NY 10010

INTRODUCTION

Let the nondeficient natural number n satisfy

(1) $$f(n) = t,$$

where $f(n) = \sigma(n) - 2n$, and t is a given nonnegative integer. Clearly, (1) is equivalent to

(1*) $$\sigma(n) = 2n + t.$$

Definition 1: m is acceptable with respect to n if m is a nondeficient proper divisor of n.

Definition 2: n is primitive if no number is acceptable with respect to n; otherwise, n is nonprimitive.

Remark 1: Primitive nondeficient numbers were defined by L. E. Dickson [3], p. 413.

If $t = 0$ in (1), then n is called perfect. It is known that when n is perfect:

(a) if n is even, then $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime (Euclid-Euler);
(b) if n is odd, then n has at least 8 distinct prime factors [4] and exceeds 10^{50} [5];
(c) n is primitive.

If $t = 1$ in (1), then n is called quasiperfect [2]. It is known that if n is quasiperfect, then:

(a) n is odd and primitive [2];
(b) n has at least 6 distinct prime factors and exceeds 10^{30} [6].

On the other hand, for $t = 3$, by inspection we obtain the nonprimitive solution $n = 18$. This suggests that nonprimitive solutions of (1), when they exist, are more easily obtained than primitive ones.

In this article, we shall determine the set of all nonprimitive solutions of (1) for each t such that $2 \leq t \leq 100$. Theorem 1 states that Table 5 contains all such solutions for the given range of values of t.

* * *

Definition 3: For given nonnegative t, let $S(t)$ denote the set of all nonprimitive solutions of (1).

Pomerance [7] showed that $S(t)$ is finite unless there exists k such that $t = \sigma(k) = 2k$.

Remark 2: In this case, a subset of $S(t)$ consists of all numbers kq where q is prime and $(k, q) = 1$. If also k is even, so that $t = 2^p(2^p - 1)$ and $2^p - 1$ is prime, then it is easily verified that $2^{2p-1}(2^p - 1)$ and $2^{p-1}(2^p - 1)^3$ also belong to $S(t)$.

Lemma 1: If m is acceptable with respect to n, then $f(m) < f(n)$.

Proof: By [7], Lemma 5, we have $\sigma(m)/m < \sigma(n)/n$. Therefore,

$$(\sigma(m) - 2m)/m < (\sigma(n) - 2n)/n,$$

i.e., $f(m)/m < f(n)/n$. Now,

$$f(m) \geq 0 \Rightarrow f(m)/n < f(m)/m \Rightarrow f(m)/n < f(n)/n \Rightarrow f(m) < f(n).$$

Definition 4: m is maximal with respect to n if m is the largest number that is acceptable with respect to n.

Lemma 2: If n is nonprimitive and m is maximal with respect to n, then there exists a prime, p, such that $n = mp$.

Proof: Let p be a prime which divides n/m, i.e., mp divides n. Now $mp > m$, so that, by hypothesis and Lemma 1, we have

$$f(mp) > f(m) \geq 0.$$

Since m is maximal with respect to n, mp is not a proper divisor of n. Thus, $mp = n$.

Corollary 2.1: m is maximal with respect to n if and only if $m = n/p$, where p is the least prime such that n/p is an integer which is acceptable with respect to n.

Proof: The proof follows directly from Lemma 2.

Corollary 2.2: If $n/2$ is a nondeficient integer, then $n/2$ is maximal with respect to n.

Proof: The proof follows directly from Corollary 2.1.

In order to construct Table 5, we first determine all nonprimitive n such that $f(n) \leq 100$. Assume, furthermore, that $n = mp$ where p is prime and m is maximal with respect to n. The need for the latter condition will be justified below.

Case 1. Suppose $(m, p) = 1$. Then

$$f(n) = f(mp) = \sigma(mp) - 2mp = (p + 1)\sigma(m) - 2mp = pf(m) + \sigma(m).$$

Thus, $2m \leq \sigma(m) \leq f(n) \leq 100$, so that $m \leq 50$. Now,

$$f(m) \geq 0 \Rightarrow m \in \{6, 12, 18, 20, 24, 28, 30, 36, 40, 42, 48\}.$$

Suppose that $m = 2^a 3^b c > 6$, where a, b, and c are natural numbers and $(6, c) = 1$. Then $n = 2^a 3^b cp$, with $(6c, p) = 1$. If $c = 1$, then $a > 1$ or $b > 1$. If $a > 1$, then $2^{a-1}3^b p$ is acceptable with respect to n, so that $2^{a-1}3^b p < 2^a 3^b$, which implies $p < 2$, an impossibility. Similarly, $b > 1$ implies $p < 3$. If $c > 1$, then $2^a 3^b p$ is acceptable with respect to n, so that $2^a 3^b p < 2^a 3^b c$, and $p < c$. Now,

$$(6c,\ p) = 1 \Rightarrow p \geq 5 \Rightarrow c \geq 6.$$

But $6c \leq m \leq 50 \Rightarrow c \leq 8$. Thus,

$$(6,\ c) = 1 \Rightarrow c = 7 \Rightarrow m = 42 \Rightarrow f(n) = 12p + 96 \geq 156,$$

contradicting the hypothesis. Likewise, $m = 40 \Rightarrow f(n) = 10p + 90 \geq 120$. If $m = 6$ and $p \geq 5$, then $f(6p) = 12$. By Corollary 2.1, it is easily verified that 6 is maximal with respect to $6p$. If $m = 28$ and $(14, p) = 1$, then $f(28p) = 56$. If $p < 11$, then $14p$ is maximal with respect to $28p$; if $p \geq 11$, then 28 is maximal with respect to $28p$. If $m = 20$ and $(10, p) = 1$, then $f(20p) = 42 + 2p$. As above, 20 is maximal with respect to $20p$ if and only if $p \geq 11$. Also,

$$f(n) = f(20p) \leq 100 \Rightarrow p \leq 29.$$

For each $m \in \{6, 28, 20\}$, and for each prime p such that m is maximal with respect to $n = mp$, and $f(n) \leq 100$, we list m, p, n, and $f(n)$ in Table 1.

TABLE 1

m	p	n	$f(n)$
6	≥ 5	$6p$	12
28	≥ 11	$28p$	56
20	11	220	64
20	13	260	68
20	17	340	76
20	19	380	80
20	23	460	88
20	29	580	100

Case 2. Suppose p divides m. Let $m = p^k r$, $n = p^{k+1} r$, where $(p, r) = 1$. Now,

$$\begin{aligned} f(m) &= \sigma(m) - 2m = \sigma(p^k r) - 2p^k r = \sigma(p^k)\sigma(r) - 2p^k r \\ &= (p^k + \sigma(p^{k-1}))\sigma(r) - 2p^k r = p^k(\sigma(r) - 2r) + \sigma(p^{k-1})\sigma(r). \end{aligned}$$

Similarly,

$$f(n) = p^{k+1}(\sigma(r) - 2r) + \sigma(p^k)\sigma(r).$$

Therefore,

$$f(n) - f(m) = (p^{k+1} - p^k)(\sigma(r) - 2r) + p^k\sigma(r) = p^k(p\sigma(r) - (p - 1)2r).$$

Now,

$$f(n) = t \Rightarrow 0 \leq f(n) - f(m) = d \leq t.$$

Therefore, the solutions of (1) may be found among the solutions of

(2) $$f(n) - f(m) = d, \text{ where } d \leq 100.$$

Let $h(p, k, r) = p^k(p\sigma(r) - (p - 1)2r)$. Then (2) is equivalent to

(3) $$h(p, k, r) = d,$$

with the restriction that

(4) $$f(p^k r) \geq 0.$$

Furthermore, (4) implies

(5) $$r \geq 2,$$

since $f(p^k) < 0$ for all primes p and all exponents k. Henceforth we consider (3).

Definition 5: Let $g(r) = \sigma(r) - r$, where r is a natural number.

Lemma 3: If

(6) $$h(2, k, r) = d, \text{ where } r \text{ is odd,}$$

then $d \equiv 0 \pmod 4$. All solutions of (6) for $d \leq 100$ are given in Table 2.

TABLE 2

d	k	$g(r)$	r	s	n	$f(n)$
4	1	1	3	+	12	4
8	1	2	*	*	*	*
8	2	1	3	+	24	12
8	2	1	5	+	40	10
8	2	1	7	+	56	8
12	1	3	*	*	*	*
16	1	4	9	+	36	19
16	2	2	*	*	*	*
16	3	1	3	+	48	28
16	3	1	5	+	80	26
16	3	1	7	+	112	24
16	3	1	11	+	176	20
16	3	1	13	+	208	18
20	1	5	*	*	*	*
24	1	6	25	–	*	*
24	2	3	*	*	*	*
28	1	7	*	*	*	*
32	1	8	49	–	*	*
32	2	4	9	+	72	51
32	3	2	*	*	*	*
32	4	1	3	+	96	60
32	4	1	5	+	160	58
32	4	1	7	+	224	56
32	4	1	11	+	352	52
32	4	1	13	+	416	50
32	4	1	17	+	544	46
32	4	1	19	+	608	44
32	4	1	23	+	736	40
32	4	1	29	+	928	34
32	4	1	31	+	992	32
36	1	9	15	+	60	48
40	1	10	*	*	*	*
40	2	5	*	*	*	*
44	1	11	21	+	84	56
48	1	12	121	–	*	*
48	2	6	25	+	200	65
48	3	3	*	*	*	*
52	1	13	27	+	108	64
52	1	13	35	+	140	56
56	1	14	169	–	*	*
56	2	7	*	*	*	*
60	1	15	33	+	132	72
64	1	16	*	*	*	*
64	2	8	49	+	392	71
64	3	4	9	+	144	115
64	4	2	*	*	*	*
64	5	1	3	+	192	124
64	5	1	5	+	320	122
64	5	1	7	+	448	120
64	5	1	11	+	704	116
64	5	1	13	+	832	114
64	5	1	17	+	1088	110
64	5	1	19	+	1216	108
64	5	1	23	+	1472	104
64	5	1	29	+	1856	98
64	5	1	31	+	1984	96
64	5	1	37	+	2368	90
64	5	1	41	+	2824	86
64	5	1	43	+	2952	84
64	5	1	47	+	3008	80
64	5	1	53	+	3392	74
64	5	1	59	+	3776	68
64	5	1	61	+	3904	66
68	1	17	39	+	156	80
68	1	17	55	–	*	*
72	1	18	289	–	*	*
72	2	9	15	+	120	120
76	1	19	65	–	*	*
76	1	19	77	–	*	*
80	1	20	361	–	*	*
80	2	10	*	*	*	*
80	3	5	*	*	*	*
84	1	21	51	+	204	96
84	1	21	91	–	*	*
88	1	22	*	*	*	*
88	2	11	21	+	168	144
92	1	23	57	+	228	104
92	1	23	85	–	*	*
96	1	24	529	–	*	*
96	2	12	121	–	*	*
96	3	6	25	+	400	161
96	4	3	*	*	*	*
100	1	25	95	–	*	*
100	1	25	119	–	*	*
100	1	25	143	–	*	*

Proof:

$$h(2, k, r) = 2^k(2\sigma(r) - 2r) = 2^{k+1}g(r) = d;\ k \geq 1 \Rightarrow d \equiv 0 \pmod 4.$$

To solve (6) for $d \leq 100$, we proceed as follows. For each d such that $d \equiv 0 \pmod 4$, and for each k such that $d \equiv 0 \pmod{2^{k+1}}$, we compute $g(r) = 2^{-(k+1)}d$. Next, we list the corresponding odd values of r, if any, using [1], Table 6.1. If no such r exists, then there is no solution of (6) corresponding to the chosen values of d and k. In this case, the r column and all columns to its right contain asterisks. For each possible r, we compute $f(2^k r)$ and list its sign, s, considering 0 to be positive. If $f(2^k r) < 0$, then there is no solution, and the last two columns contain asterisks. If $f(2^k r) \geq 0$, then we have obtained a solution of (6), and $n = 2^{k+1}r$ corresponds to a solution of (2). In this case, we list n and $f(n)$. If $g(r) = 1$, then r is prime and (4) implies $r \leq 2^k - 1$. In this case, we list only such r.

Lemma 4: If

(7) $$p\sigma(r) - (p - 1)2r = v,$$

where p is an odd prime, $(p, r) = 1$, and (4) holds, then we must have:

(8) $$\sigma(r) = pv + (p - 1)2u;$$

(9) $$r = (p + 1)v/2 + pu;$$

(10) $$(p, v) = 1;$$

(11) $$r \leq v\sigma(p^k)/2.$$

Proof: Solving (7) for $\sigma(r)$ and $2r$ in terms of p and v, one has

(8*) $$\sigma(r) = pv + (p + 1)w;$$

(9*) $$2r = (p + 1)v + pw.$$

(9*) $\Rightarrow w$ is even. Setting $w = 2u$, one obtains (8) and (9). (10) follows directly from the hypothesis. (11) is derived from (4) as follows:

$$f(p^k r) \geq 0 \Rightarrow \sigma(p^k)\sigma(r) \geq 2p^k r \Rightarrow p\sigma(p^k)\sigma(r) \geq 2p^{k+1}r;$$
$$(7) \Rightarrow p\sigma(p^k)\sigma(r) - (p - 1)\sigma(p^k)2r = v\sigma(p^k).$$

Therefore,

$$2p^{k+1}r - (p^{k+1} - 1)2r \leq v\sigma(p^k) \Rightarrow 2r \leq v\sigma(p^k) \Rightarrow r \leq v\sigma(p^k)/2.$$

Corollary 4.1: If

(12) $$h(p, k, r) = p^j s,$$

where p is an odd prime, $s \geq 1$, and $(p, s) = 1$, then $k = j$.

Proof: By hypothesis, (7) holds with $v = p^{j-k}s$. Now (10) implies $j - k = 0$, i.e., $k = j$.

Lemma 5: If

(13) $$h(p, k, r) = q,$$

where q is an odd prime, then $k = 1$, and for some integer, a, we have

$$p = q = 2^a - 1,\ r = 2^{a-1}.$$

Proof: p divides $q \Rightarrow p = q$. Hypothesis and Corollary 4.1 $\Rightarrow k = 1$. Thus, (13) reduces to (7) with $v = 1$. From (11), we have $r \leq (p + 1)/2$, so that

$u \le 0$ in (9). But (5) and (9) $\Rightarrow u \ge (3 - p)/2p$. Therefore, $u = 0$, i.e., $\sigma(r) = p$, $r = (p + 1)/2$. $\sigma(r) = p \Rightarrow r = s^{a-1}$ for some prime, s, and some integer $a \ge 2$. Now,

$$s^{a-1} + s^{a-2} + \cdots + s + 1 = \sigma(s^{a-1}) = \sigma(r) = p = 2r - 1 = 2s^{a-1} - 1.$$

Therefore, 2 divides s, i.e., $s = 2$. Thus, $r = 2^{a-1}$, $p = 2^a - 1$.

Lemma 6: For any j, the unique solution of

$$h(p, k, r) = 3^j \tag{14}$$

is: $p = 3$, $k = j$, $r = 2$.

Proof: Clearly, $p = 3$, $k = j$, and (14) reduces to $3\sigma(r) - 4r = 1$. (8) and (9) $\Rightarrow r = 2 + 3u$, $\sigma(r) = 3 + 4u \Rightarrow \sigma(r)$ is odd $\Rightarrow r = 2^a b^2$ with $a \ge 0$ and b odd. Furthermore, $(3, r) = 1 \Rightarrow (6, b) = 1$. $r \equiv 2 \pmod 3 \Rightarrow 2^a b^2 \equiv 2 \pmod 3 \Rightarrow 2^a \equiv 2 \pmod 3 \Rightarrow a \ge 1 \Rightarrow r$ is even $\Rightarrow \sigma(r)/r \ge 3/2 \Rightarrow 2\sigma(r) \ge 3 \Rightarrow 6 + 8u \ge 6 + 9u \Rightarrow u \le 0 \Rightarrow r \le 2$. By (5), $r = 2$.

Lemma 7: For no j does

$$h(p, k, r) = 5^j \tag{15}$$

have a solution.

Proof: If a solution exists, then $p = 5$, $k = j$, and (15) reduces to $5\sigma(r) - 8r = 1$, so that $r = 3 + 5u$, $\sigma(r) = 5 + 8u$, and $r = 2^a b^2$ with $a \ge 0$ and $(10, b) = 1$. Now $r \equiv 3 \pmod 5 \Rightarrow 2^a b^2 \equiv 3 \pmod 5 \Rightarrow 2^a \equiv 2$ or $3 \pmod 5 \Rightarrow a = 2c + 1$. But $\sigma(2^{2c+1}) \equiv 0 \pmod 3$. Thus,

$$\sigma(r) \equiv 0 \pmod 3 \Rightarrow u \equiv 2 \pmod 3 \Rightarrow r \equiv 1 \pmod 3$$
$$\Rightarrow 2^{2c+1} b^2 \equiv 1 \pmod 3 \Rightarrow b^2 \equiv 2 \pmod 3,$$

an impossibility.

Lemma 8: If

$$h(p, k, r) = q^j, \tag{16}$$

where q is an odd prime, $j \ge 2$, and $q^j \le 100$, then $k = j$ and either

(i) $p = 3$, $r = 2$, $2 \le j \le 4$; or

(ii) $p = 7$, $r = 4$, $j = 2$.

Proof:

$$q^2 \le q^j \le 100 \Rightarrow q \le 10 \Rightarrow q \in \{3, 5, 7\}.$$

If $q = 3$, then $3^j \le 100 \Rightarrow j \le 4$, and the solutions of (16) are given by Lemma 6. Lemma 7 $\Rightarrow q \ne 5$. If $q = 7$, then $7^j \le 100 \Rightarrow j = 2$, and (16) reduces to $7\sigma(r) - 12r = 1$. Therefore, by Lemma 4, we have

$$\sigma(r) = 7 + 12u,\ r = 4 + 7u,\ r \le 28.$$

By inspection, we must have $r = 4$.

Combining the results of Lemmas 5 and 8, we list all solutions of

$$h(p, k, r) = q^j, \tag{17}$$

with q an odd prime and $q^j \le 100$, in Table 3. For each q^j, we list p, k, r, as well as the m, n of the corresponding solution of (2), and $f(n)$. It is easily verified that in each case m is maximal with respect to n.

TABLE 3

q	p	k	r	m	n	$f(n)$
3	3	1	2	6	18	3
7	7	1	4	28	196	7
9	3	2	2	18	54	12
27	3	3	2	54	162	39
31	31	1	16	496	15736	31
49	7	2	4	196	1372	56
81	3	4	2	162	486	120

Lemma 9: For no odd prime q does

(18) $$h(p, k, r) = 2q$$

have a solution.

Proof: If a solution exists, then by hypothesis, Lemma 3, and Corollary 4.1, we have $p \neq 2$, $p = q$, and $k = 1$. Thus, (18) reduces to (7) with $v = 2$, and we have $\sigma(r) = 2p + (p - 1)2u$, $r = p + 1 + pu$, $r \leq p + 1$. Now, (5) $\Rightarrow u = 0$, $r = p + 1$, $\sigma(r) = 2p$. Let $r = 2^a b$ with $a \geq 1$ and b odd. Then,

$$\sigma(r) = \sigma(2^a)\sigma(b) = 2p, \text{ so that } \sigma(b) = 2,$$

an impossibility.

Definition 6: If $0 \leq a \leq 3$, let

$$C_a = \{r : 2 \leq r \leq 100, \text{ and } \sigma(r) \equiv a \pmod 4\}.$$

By inspection, we have

C_0 = {3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, 35, 38, 39, 42, 43, 44, 46, 47, 48, 51, 54, 55, 56, 57, 59, 60, 62, 63, 65, 66, 67, 69, 70, 71, 75, 76, 77, 78, 79, 83, 84, 85, 86, 87, 88, 91, 92, 93, 94, 95, 96, 99};

C_1 = {9, 49, 50, 81, 100};

C_2 = {5, 10, 13, 17, 20, 26, 29, 34, 37, 40, 41, 45, 52, 53, 58, 61, 68, 73, 74, 80, 82, 89, 90, 97};

C_3 = {2, 4, 8, 16, 18, 25, 32, 36, 64, 72, 98}.

Lemma 10: In (3), if $r = q^b$, where q is prime, then $q = 2$ and $r \in C_3$.

Proof: (4) implies

$$(p/(p - 1))(q/(q - 1)) > \sigma(p^k r)/p^k r \geq 2 \Rightarrow q < 2(p - 1)/(p - 2).$$

If $p = 3$, then $q < 4 \Rightarrow q = 2$, since $(p, r) = 1$. If $p \geq 5$, then $q < 8/3 \Rightarrow q = 2$. $\sigma(2^b) = 2^{b+1} - 1 \equiv 3 \pmod 4 \Rightarrow r \in C_3$.

Lemma 11: All solutions of (3) such that p is odd, $d \leq 100$, $d \neq q^j$, where q is an odd prime, are given in Table 4.

Proof: To obtain the desired solutions of (3), we proceed as follows: for each $d \neq 2q$, $\neq q^j$, for each odd prime p such that $p^k v = d$, $(p, v) = 1$, we list p, k, v. If r exists such that (7) holds, we must have:

(i) $r \leq \underline{r} = [v\sigma(p^k)/2]$;

(ii) $r \equiv v(p + 1)/2 \pmod p$;

(iii) $r \in C_a$, where $pv \equiv a \pmod 4$;

(iv) r is not a power of a prime unless $r = 2^b$ and $a = 3$.

For convenience, we list $\underline{r}$, r_p [the least positive residue (mod p) of $v(p+1)/2$], and a. If no r exists satisfying the above conditions, then (3) has no solution corresponding to that particular choice of p, d. In this case, the r column and all remaining columns contain asterisks. For each r which does satisfy the conditions, we compute and list $w = p\sigma(r) - (p - 1)2r$. If $w \neq v$, then we have no solution, and the remaining columns contain asterisks. If $w = v$, we have a solution. We list the values m and n of the corresponding solution of (2). Finally, we test m for maximality with respect to n using Corollaries 2.1 and 2.2. If the test is positive, the max column says yes and the final column lists $f(n)$; otherwise, the max column says no and the final column contains an asterisk.

TABLE 4

d	p	k	v	$\underline{r}$	r_p	a	r	w	m	n	max	$f(n)$
12	3	1	4	8	2	0	*	*	*	*	*	*
15	3	1	5	10	1	3	4	5	12	36	no	*
15	5	1	3	9	4	3	4	3	20	100	yes	17
18	3	2	2	13	1	2	10	14	*	*	*	*
20	5	1	4	12	2	0	12	44	*	*	*	*
21	3	1	7	14	2	1	9	3	*	*	*	*
21	7	1	3	12	5	1	9	−17	*	*	*	*
24	3	1	8	16	1	0	*	*	*	*	*	*
28	7	1	4	16	2	0	*	*	*	*	*	*
30	3	1	10	20	2	2	20	46	*	*	*	*
30	5	1	6	18	3	2	18	51	*	*	*	*
33	3	1	11	22	1	1	*	*	*	*	*	*
33	11	1	3	18	7	1	*	*	*	*	*	*
35	5	1	7	21	1	3	16	27	*	*	*	*
35	7	1	5	20	6	3	*	*	*	*	*	*
36	3	2	4	26	2	0	14	129	*	*	*	*
39	3	1	13	26	2	3	8	13	24	72	no	*
39	13	1	3	21	8	3	8	3	104	1352	yes	41
40	5	1	8	24	4	0	14	8	70	350	yes	44
42	3	1	14	28	1	2	10	14	30	90	yes	54
42	7	1	6	24	3	2	10	6	70	490	yes	46
44	11	1	4	24	2	0	24	180	*	*	*	*
45	3	2	5	32	1	3	4	5	36	108	no	*
45	3	2	5	32	1	3	16	29	*	*	*	*
45	3	2	5	32	1	3	25	−7	*	*	*	*
45	5	1	9	27	2	1	*	*	*	*	*	*
48	3	1	16	32	2	0	14	16	42	126	yes	60
50	5	2	2	31	1	2	26	2	650	3250	yes	52
51	3	1	17	34	1	3	4	5	*	*	*	*
51	3	1	17	34	1	3	16	29	*	*	*	*
51	3	1	17	34	1	3	25	−7	*	*	*	*
51	17	1	3	27	10	3	*	*	*	*	*	*
52	13	1	4	28	2	0	15	−48	*	*	*	*
52	13	1	4	28	2	0	28	56	*	*	*	*
54	3	3	2	40	1	2	10	14	*	*	*	*

d	p	k	v	$\underline{r}$	r	a	r	w	m	n	max	$f(n)$
54	3	3	2	40	1	2	34	26	*	*	*	*
54	3	3	2	40	1	2	40	110	*	*	*	*
55	5	1	11	33	3	1	8	11	40	200	no	*
55	5	1	11	33	3	1	18	51	*	*	*	*
55	11	1	5	30	8	1	8	5	88	968	yes	59
56	7	1	8	32	4	0	*	*	*	*	*	*
57	3	1	19	38	2	1	*	*	*	*	*	*
57	19	1	3	30	11	1	*	*	*	*	*	*
60	3	1	20	40	1	0	22	20	66	198	yes	72
60	5	1	12	36	1	0	6	12	30	150	yes	72
63	3	2	7	45	2	1	*	*	*	*	*	*
63	7	1	9	36	1	3	8	9	56	392	no	*
65	5	1	13	39	4	1	*	*	*	*	*	*
65	13	1	5	35	9	1	9	-47	*	*	*	*
66	3	1	22	44	2	2	20	46	*	*	*	*
66	3	1	22	44	2	2	26	22	78	234	yes	78
66	11	1	6	36	3	2	*	*	*	*	*	*
68	17	1	4	36	2	0	*	*	*	*	*	*
69	3	1	23	46	1	1	*	*	*	*	*	*
69	23	1	3	36	13	1	*	*	*	*	*	*
70	5	1	14	42	2	2	*	*	*	*	*	*
70	7	1	10	40	5	2	26	-18	*	*	*	*
70	7	1	10	40	5	2	40	150	*	*	*	*
72	3	2	8	52	1	0	22	20	*	*	*	*
72	3	2	8	52	1	0	28	56	*	*	*	*
72	3	2	8	52	1	0	46	32	*	*	*	*
75	3	1	25	50	2	3	8	13	*	*	*	*
75	3	1	25	50	2	3	32	61	*	*	*	*
75	5	2	3	46	4	3	4	3	100	500	yes	92
76	19	1	4	40	2	0	21	-148	*	*	*	*
77	7	1	11	44	2	1	9	-17	*	*	*	*
77	11	1	7	42	9	1	9	-37	*	*	*	*
78	3	1	26	52	1	2	10	14	*	*	*	*
78	3	1	26	52	1	2	34	26	102	306	yes	90
78	3	1	26	52	1	2	40	110	*	*	*	*
78	3	1	26	52	1	2	52	86	*	*	*	*
78	13	1	6	42	3	2	*	*	*	*	*	*
80	5	1	16	48	3	0	28	56	*	*	*	*
80	5	1	16	48	3	0	33	-24	*	*	*	*
80	5	1	16	48	3	0	38	-4	*	*	*	*
80	5	1	16	48	3	0	48	296	*	*	*	*
84	3	1	28	56	2	0	38	28	114	342	yes	96
84	7	1	12	48	6	0	6	12	42	294	yes	96
84	7	1	12	48	6	0	27	-44	*	*	*	*
85	5	1	17	51	1	1	*	*	*	*	*	*
85	17	1	5	45	11	1	*	*	*	*	*	*
87	3	1	29	58	1	3	4	5	*	*	*	*
87	3	1	29	58	1	3	16	29	48	144	no	*
87	3	1	29	58	1	3	25	-7	*	*	*	*
87	29	1	3	45	16	3	16	3	474	13456	yes	89
88	11	1	8	48	4	0	15	-36	*	*	*	*

d	p	k	v	$\underline{r}$	r	a	r	w	m	n	max	$f(n)$
88	11	1	8	48	4	0	48	404	*	*	*	*
90	3	2	10	65	2	2	20	46	*	*	*	*
90	3	2	10	65	2	2	26	22	*	*	*	*
90	5	1	18	54	4	2	*	*	*	*	*	*
91	7	1	13	52	3	3	*	*	*	*	*	*
91	13	1	7	49	10	3	36	319	*	*	*	*
92	23	1	4	48	2	0	48	80	*	*	*	*
93	3	1	31	62	2	1	50	79	*	*	*	*
93	31	1	3	48	17	1	*	*	*	*	*	*
95	5	1	19	57	2	3	32	59	*	*	*	*
95	19	1	5	50	12	3	*	*	*	*	*	*
96	3	1	32	64	1	0	22	20	*	*	*	*
96	3	1	32	64	1	0	28	56	*	*	*	*
96	3	1	32	64	1	0	46	32	138	414	yes	108
96	3	1	32	64	1	0	55	-4	*	*	*	*
98	7	2	2	57	1	2	*	*	*	*	*	*
99	3	2	11	71	2	1	50	79	*	*	*	*
99	11	1	9	54	10	3	32	53	*	*	*	*
100	5	2	4	62	2	0	12	44	*	*	*	*
100	5	2	4	62	2	0	22	4	550	2750	yes	116
100	5	2	4	62	2	0	27	-16	*	*	*	*
100	5	2	4	62	2	0	42	144	*	*	*	*
100	5	2	4	62	2	0	57	-76	*	*	*	*
100	5	2	4	62	2	0	62	-6	*	*	*	*

Combining the results of Tables 1, 2, 3, and 4, we form Table 5. For each t such that $2 \leq t \leq 100$ and $S(t)$ is nonempty, we list the members of $S(t)$. If $S(t)$ is empty, then t does not appear as an entry. The requirement that the solutions listed in Tables 1, 2, 3, and 4 satisfy a maximality condition assures that distinct entries from these tables yield distinct corresponding entries in Table 5. Therefore, we have proved:

Theorem 1: All solutions of (1) such that n is nonprimitive and $2 \leq t \leq 100$ are given in Table 5.

TABLE 5

t	$S(t)$	t	$S(t)$	t	$S(t)$	t	$S(t)$
3	18	28	48	52	352, 3250	74	3392
4	12	31	15736	54	90	76	340
7	196	32	992	56	224, 1372, $28p$**	78	234
8	56	34	928	58	160	80	156, 380, 3008
10	40	39	162	59	968	84	2952
12	24, 54, $6p$*	40	736	60	96, 126	86	2824
17	100	41	1352	64	108, 220	88	460
18	208	44	350, 608	65	200	89	13456
19	36	46	490, 544	66	3904	90	306, 2368
20	176	48	60	68	260, 3776	92	500
24	112	50	416	71	392	96	204, 294, 342, 1984
26	80	51	72	72	132, 150, 198	98	1856
						100	580

*p prime, $(6, p) = 1$ **p prime, $(14, p) = 1$

REFERENCES

1. J. Alanen. "Empirical Study of Aliquot Series." (Unpublished doctoral dissertation, Yale University, 1972.)
2. P. Cattaneo. "Sui numeri quasiperfetti." *Boll. Un. Mat. Ital.* (3) 6 (1951):59-62.
3. L. E. Dickson. "Finiteness of the Odd Perfect and Primitive Abundant Numbers With n Distinct Factors." *Amer. J. Math.* 35 (1913):413-422.
4. P. Hagis, Jr. "Every odd Perfect Number Has at Least Eight Distinct Prime Factors." *Notices, Amer. Math. Society* 22, No. 1 (1975):Ab. 720-10-14.
5. P. Hagis, Jr. "A Lower Bound for the Set of Odd Perfect Numbers." *Math. Comp.* 27, No. 124 (1973).
6. M. Kishore. "Quasiperfect Numbers Have at Least Six Distinct Prime Factors." *Notices, Amer. Math. Society* 22, No. 4 (1975):Ab. 75T-A113.
7. C. Pomerance. "On the Congruences $\sigma(n) \equiv a \pmod{n}$ and $n \equiv a \pmod{\phi(n)}$." *Acta Arith.* 26 (1974):265-272,

*****

WEIGHTED STIRLING NUMBERS OF THE FIRST AND SECOND KIND—I

L. Carlitz

Duke University, Durham, N.C. 27706

1. INTRODUCTION

The Stirling numbers of the first and second kind can be defined by

(1.1)
$$(x)_n \equiv x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} S_1(n, k)x^k$$

and

(1.2)
$$x^n = \sum_{k=0}^{n} S(n, k)x(x-1)\cdots(x-k+1),$$

respectively.

It is well known that $S_1(n, k)$ is the number of permutations of

$$Z_n = \{1, 2, \ldots, n\}$$

with k cycles and that $S(n, k)$ is the number of partitions of the set Z_n into k blocks [1, Ch. 5], [2, Ch. 4]. These combinatorial interpretations suggest the following extensions.

Let n, k be positive integers, $n \geq k$, and let $k_1, k_2, \ldots, k$ be nonnegative integers such that

(1.3)
$$\begin{cases} k = k_1 + k_2 + \cdots + k_n \\ n = k_1 + 2k_2 + \cdots + nk_n. \end{cases}$$

We define $\overline{S}(n, k, \lambda)$, $\overline{S}_1(n, k, \lambda)$, where λ is a parameter, in the following way.

(1.4)
$$\overline{S}(n, k, \lambda) = \sum\sum(k_1\lambda + k_2\lambda^2 + \cdots + k_n\lambda^n),$$

where the inner summation is over all partitions of Z_n into k_1 blocks of cardinality 1, k_2 blocks of cardinality 2, ..., k_n blocks of cardinality n; the outer summation is over all $k_1, k_2, \ldots, k_n$ satisfying (1.3).