these cannot vanish. Therefore,

$$C_{n-2} \neq 0.$$

So, $C_0 = 0$ implies

$$xR_{n-1} = C_n R_n + C_{n-1}R_{n-1} + C_{n-2}R_{n-2}.$$

Hence, by Favard's Theorem, these R_n must be an orthogonal set with respect to some weighting function $w_1(x)$ and some range $[c, d]$ if the integral be considered a Stieltjes integral.

If $C_0 \neq 0$, the R_n do not satisfy a three-term recursion formula (unless $n = 2$) and by applying the contrapositive of the converse, we see that the R_n cannot be an orthogonal set with respect to any weighting function and range.

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ON SOME SYSTEMS OF DIOPHANTINE EQUATIONS INCLUDING THE ALGEBRAIC SUM OF TRIANGULAR NUMBERS

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The natural number of the form

$$t_n = \binom{n+1}{2} = \frac{1}{2}n(n+1),$$

where n is a natural number, is referred to as the nth triangular number. The aim of this work is to give solutions of some equations and systems of equations in triangular numbers.

1. THE EQUATION $t_{t_x} + t_{t_y} = t_{t_z}$

It is well known that the equation

$$t_x + t_y = t_z \tag{1}$$

has infinitely many solutions in triangular numbers t_x, t_y, and t_z. For example, it follows immediately from the formula:

$$t_{(2n+1)k} + t_{4t_n k+n} = t_{(4t_n+1)k+n}. \tag{2}$$

We can ask whether there exists a solution of the equation:

$$t_{t_x} + t_{t_y} = t_{t_z}. \tag{3}$$

The answer to this question is positive, because there exist two solutions:

$$t_{t_{59}} + t_{t_{77}} = t_{t_{83}} \quad \text{and} \quad t_{t_{104}} + t_{t_{213}} = t_{t_{216}}.$$

The problem of finding all the solutions to equation (1) above will be solved in a subsequent paper.

2. TRIPLES OF TRIANGULAR NUMBERS, THE SUMS OR DIFFERENCES OF ANY TWO OF WHICH ARE ALSO TRIANGULAR NUMBERS

The system of three equations:

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y + 2t_z &= t_q, \end{aligned} \tag{4}$$

has infinitely many solutions in triangular numbers t_x, t_y, t_z, t_u, t_v, and t_q. This theorem can be proved by insertion of the following formulas into equations (4):

$$\begin{aligned} &x = n, \quad y = \frac{1}{2}(t_n - 3), \quad z = t_n - 1, \\ &u = \frac{1}{2}(t_n + 1), \quad v = t_n, \quad q = \frac{3}{2}(t_n - 1), \end{aligned} \tag{5}$$

where n is a natural number of the form $4k + 1$ or $4k + 2$ for natural k.

In particular, putting $n = 14$, we have:

$$\begin{aligned} &x = 14, \quad y = 51, \quad z = 104, \\ &u = 53, \quad v = 105, \quad q = 156. \end{aligned}$$

Since $t_q - t_z = t_{156} - t_{104} = t_{116} = t_w$, we obtain a solution of the system of equations:

$$\begin{aligned} t_x + t_y &= t_u, &\text{in the numbers: } t_{14} + t_{51} &= t_{53}, \\ t_x + t_z &= t_v, & t_{14} + t_{104} &= t_{105}, \\ t_y + t_z &= t_w, & t_{51} + t_{104} &= t_{116}. \end{aligned} \tag{6}$$

We see that there exists a triple of triangular numbers whose sums in pairs are also triangular numbers. The problem of whether there exist three different triangular numbers, the sum of any two of which is a triangular number was formulated by W. Sierpiński [1].

<u>*Theorem*</u>: Suppose that $x > y > z$; then each of the systems of equations:

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y + t_z &= t_w; \end{aligned} \tag{7.1}$$

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y - t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v; \end{aligned} \tag{7.2}$$

$$\begin{aligned} t_x + t_y &= t_u, \\ t_x - t_z &= t_v, \\ t_y + t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v; \end{aligned} \tag{7.3}$$

$$(7.4)\qquad \begin{aligned} t_x - t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y + t_z &= t_w, \quad \text{where } x \neq w,\ z \neq u; \end{aligned}$$

$$(7.5)\qquad \begin{aligned} t_x - t_y &= t_u, \\ t_x - t_z &= t_v, \\ t_y + t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v,\ z \neq u; \end{aligned}$$

$$(7.6)\qquad \begin{aligned} t_x - t_y &= t_u, \\ t_x + t_z &= t_v, \\ t_y - t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v; \end{aligned}$$

$$(7.7)\qquad \begin{aligned} t_x + t_y &= t_u, \\ t_x - t_z &= t_v, \\ t_y - t_z &= t_w; \end{aligned}$$

$$(7.8)\qquad \begin{aligned} t_x - t_y &= t_u, \\ t_x - t_z &= t_v, \\ t_y - t_z &= t_w, \quad \text{where } x \neq w,\ y \neq v,\ z \neq u; \end{aligned}$$

has infinitely many solutions in triangular numbers t_x, t_y, t_z, t_u, t_v, and t_w.

Proof: We prove even more. Each of the following systems of equations has infinitely many solutions in natural numbers x and y.

$$(8.1)\qquad \begin{aligned} t_{16x+2} + t_{12x+2} &= t_{20x+3}, \\ t_{16x+2} + t_{9x+2} &= t_y, \\ t_{12x+2} + t_{9x+2} &= t_{15x+3}; \end{aligned}$$

$$(8.2)\qquad \begin{aligned} t_{16x+2} + t_{13x+2} &= t_y, \\ t_{16x+2} + t_{12x+2} &= t_{20x+3}, \\ t_{13x+2} - t_{12x+2} &= t_{5x}; \end{aligned}$$

$$(8.3)\qquad \begin{aligned} t_{16x+2} + t_{12x+2} &= t_{20x+3}, \\ t_{16x+2} - t_{9x+2} &= t_y, \\ t_{12x+2} + t_{9x+2} &= t_{15x+3}; \end{aligned}$$

$$(8.4)\qquad \begin{aligned} t_{15x+3} - t_{12x+2} &= t_{9x+2}, \\ t_{15x+3} + t_{5x} &= t_y, \\ t_{12x+2} + t_{5x} &= t_{13x+2}, \end{aligned} \quad \text{or} \quad \begin{aligned} t_{13x+2} - t_{12x+2} &= t_{5x}, \\ t_{13x+2} + t_{9x+2} &= t_y, \\ t_{12x+2} + t_{9x+2} &= t_{15x+3}; \end{aligned}$$

$$(8.5)\qquad \begin{aligned} t_{15x+3} - t_{12x+2} &= t_{9x+2},\\ t_{15x+3} - t_{5x} &= t_y,\\ t_{12x+2} + t_{5x} &= t_{13x+2}; \end{aligned}$$

$$(8.6)\qquad \begin{aligned} t_{16x+2} - t_{13x+2} &= t_y,\\ t_{16x+2} + t_{12x+2} &= t_{20x+3},\\ t_{13x+2} - t_{12x+2} &= t_{5x}; \end{aligned}$$

$$(8.7)\qquad \begin{aligned} t_{52x+2} + t_{39x+2} &= t_{65x+3},\\ t_{52x+2} - t_{36x+2} &= t_y,\\ t_{39x+2} - t_{36x+2} &= t_{15x}; \end{aligned}$$

$$(8.8)\qquad \begin{aligned} t_{15x+3} - t_{13x+2} &= t_y,\\ t_{15x+3} - t_{12x+2} &= t_{9x+2},\\ t_{13x+2} - t_{12x+2} &= t_{5x}. \end{aligned}$$

The systems of equations (8.1)-(8.8) are, respectively, equivalent to the following equations, for which there exist initial solutions given below:

$$(9.1)\qquad 337x^2 + 125x - y^2 - y = -12,\quad x_0 = 0,\quad y_0 = 3;$$
$$(9.2)\qquad 425x^2 + 145x - y^2 - y = -12,\quad x_0 = 0,\quad y_0 = 3;$$
$$(9.3)\qquad 175x^2 + 35x - y^2 - y = 0,\quad x_0 = 0,\quad y_0 = 0;$$
$$(9.4)\qquad 250x^2 + 110x - y^2 - y = -12,\quad x_0 = 0,\quad y_0 = 3;$$
$$(9.5)\qquad 200x^2 + 100x - y^2 - y = -12,\quad x_0 = 0,\quad y_0 = 3;$$
$$(9.6)\qquad 87x^2 + 15x - y^2 - y = 0,\quad x_0 = 0,\quad y_0 = 0;$$
$$(9.7)\qquad 1408x^2 + 80x - y^2 - y = 0,\quad x_0 = 0,\quad y_0 = 0;$$
$$(9.8)\qquad 56x^2 + 40x - y^2 - y = -6,\quad x_0 = 0,\quad y_0 = 2.$$

From the theory of Pell's equation (also referred to as Fermat's equation), it follows that if, simultaneously, k and m are natural numbers, 1, n, and q are integers, then the product $k \cdot m$ is not a square, and if there exists an initial solution of the equation,

$$(10)\qquad kx^2 + 1x - my^2 - ny = q,$$

in integers x_0 and y_0, where $\left(x_0 + \frac{1}{2k}\right)^2 + \left(y_0 + \frac{n}{2m}\right)^2 \neq 0$, then equation (10) has infinitely many solutions in natural numbers x and y. Applying this to equations (9.1)-(9.8) we prove that all the systems of equations (8.1)-(8.8) have infinitely many solutions in natural numbers x and y. This theorem is thus proved.

Some years ago, A. Schinzel found the following proof for the statement that there exist infinitely many triples of different triangular numbers for which the sum of any two is a triangular number [private communication from A. Schinzel].

Schinzel's Proof (unpublished): It is well known that the equation

$$x^2 - 424y^2 = 1$$

has infinitely many solutions, where $x \equiv 1 \pmod{106}$ [in every solution, we have $\pm x \equiv 1 \pmod{106}$]. Putting

$$k = 5y - \frac{25}{106}(x - 1) - 1,$$
$$1 = \frac{5}{2}(x - 1) - 50y + 2,$$

we find

$$t_{5k+4} + t_{9k+6} = t_1,$$
$$t_{5k+4} + t_{12k+9} = t_{13k+10},$$
$$t_{9k+6} + t_{12k+9} = t_{15k+11}.$$

3. SYSTEMS OF EQUATIONS INCLUDING THE ALGEBRAIC SUM AND THE PRODUCT OF TRIANGULAR NUMBERS

W. Sierpiński [1] has asked whether there exists a pair of triangular numbers such that the sum and the product of these numbers are triangular numbers. We have found some such systems of equations for which there exist one or two solutions in triangular numbers, e.g.:

(11) 1. $$t_x - t_y = t_u, \quad t_x + t_y = t_v, \quad t_x t_y = t_w,$$
$$t_{18} - t_{14} = t_{11}, \quad t_{18} + t_{14} = t_{23}, \quad t_{18}t_{14} = t_{189}.$$

(This solution was found by K. Szymiczek [2].)

(12) 2. $$t_x + t_y = t_u, \quad t_x t_y = t_v, \quad (t_x + 1)t_y = t_w,$$
$$t_9 + t_{13} = t_{16}, \quad t_9 t_{13} = t_{90}, \quad (t_9 + 1)t_{13} = t_{91}.$$

(13) 3. $$t_x - t_y = t_u, \quad t_x t_y = t_v, \quad t_x/t_y - 1 = t_w,$$
$$t_{21} - t_6 = t_{20}, \quad t_{21}t_6 = t_{98}, \quad t_{21}/t_6 - 1 = t_4.$$

(14) 4. $$t_x - t_y = t_q, \quad t_x + t_z = t_u,$$
$$t_x t_y = t_v, \quad t_x t_z = t_w,$$

$$t_{21} - t_6 = t_{20}, \quad t_{21} + t_{35} = t_{41},$$
$$t_{21}t_6 = t_{98}, \quad t_{21}t_{35} = t_{539},$$

and

$$t_{63} - t_{38} = t_{50}, \quad t_{63} + t_{219} = t_{228},$$
$$t_{63}t_{38} = t_{1728}, \quad t_{63}t_{219} = t_{9855}.$$

(15) 5. $$t_x + t_z = t_q, \quad t_y - t_z = t_u,$$
$$t_x t_z = t_v, \quad t_y t_z = t_w,$$

(continued)

5. *continued*

$$t_{29} + t_{69} = t_{75}, \quad t_{168} - t_{69} = t_{153},$$
$$t_{29}t_{69} = t_{1449}, \quad t_{168}t_{69} = t_{8280}.$$

6. For the system of equations,

$$t_x + t_y = t_u, \qquad t_xt_y = t_v, \tag{16}$$

there exists also the solution:

$$t_{505} + t_{531} = t_{733}, \quad t_{505}t_{531} = t_{189980}.$$

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ON EULER'S SOLUTION TO A PROBLEM OF DIOPHANTUS—II

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1. INTRODUCTION

In an earlier paper [1] we considered solutions to a system of equations:

$$x_ix_j + 1 = y_{ij}^2; \quad 1 \le i < j \le n.$$

In this note we look at the generalized problems:

$$x_ix_j + a = y_{ij}^2, \quad a \ne 0. \tag{1.1}$$

In Section 2 we apply the results of [1] to the solutions of (1.1). In Section 3 we consider the following problem: Find $n \times 2$ matrices

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

so that $a_ib_j \pm a_jb_i = \pm 1$ for all $1 \le i < j \le n$. In Section 4 we apply the results of Section 3 to get two-parameter families of solutions of (1.1), linear in a, for $n = 4$.

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