If $j = 2$ in (C6), we obtain

(3.7) $$a_{001}^2 + a_{011}^2 + a_{001}a_{011} = 1.$$

Using (3.6) and (3.7) along with the fact that $a_{001} = 1$, we see that $a_{011} = 0$.

Since all the variables in (C7) have already been uniquely determined, we proceed to (C8), where we obtain

(3.8) $$a_{001}^2 + a_{101}^2 + 2a_{001}a_{101} = 1$$

and

(3.9) $$a_{001}^2 + a_{101}^2 + a_{001}a_{101} = 1,$$

so that $a_{101} = 0$.

From (C10), we obtain

(3.10) $$a_{010}^2 + a_{110}^2 + 2a_{010}a_{110} = 1$$

and

(3.11) $$a_{010}^2 + a_{110}^2 + a_{010}a_{110} = 1,$$

so that $a_{110} = 0$.

From (C12), we obtain, after simplification,

(3.12) $$a_{111}^2 + 2a_{111} = 0$$

and

(3.13) $$a_{111}^2 + a_{111} = 0,$$

so that $a_{111} = 0$.

We have now uniquely determined all 27 coefficients in (3.1). Thus,

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

is the only reduced local permutation polynomial in three variables over Z_3 and, hence, there is precisely one reduced Latin cube of order three. If we list the cube in terms of the three Latin squares of order three which form its different levels, we can list the only reduced Latin cube of order three as

012	120	201
120	201	012
201	012	120.

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SOME COMBINATORIAL IDENTITIES

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In this paper, we wish to derive some combinatorial identities (partly known, partly apparently new) by combining well-known recurrence relations with known forms for characteristic polynomials of paths and cycles (i.e., of their adjacency matrices). We also obtain some extensions of known results.

Define $P_0 = 1$, $P_1 = x$. For $n > 1$, define

$$P_n = P_n(x) = xP_{n-1} - P_{n-2}. \tag{1}$$

This recurrence relation has been investigated by Liebestruth [5] (see also [2, v. I, p. 402]). The formula for P_n is given as

$$P_n = \sum_{k=0} (-1)^k \binom{n-k}{k} x^{n-2k} \tag{2}$$

for every nonnegative integer n.

The following Fibonacci polynomial is treated in [8].

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

(see also [4]). The polynomial P_n is hence essentially a Fibonacci polynomial. As such (2) appears as a problem in [7]. The connection between the polynomial P_n and F_n is easily seen to be

$$P_n(x) = i^n F_{n+1}(-ix), \tag{3}$$

where i is the imaginary unit.

By postulating $P_{-1} = 0$, and in general

$$P_{-n} = -P_{n-2} \tag{4}$$

for all positive integers n, P_n turns out to be a polynomial for every integer n. It is easy to check that both (1) and (4) are valid for all integral n.

Using (1) and the induction principle, we can show that, for $n \geq 0$, we have

$$x^n = \sum_{k=0} \binom{n-1}{k} P_{n-2k}. \tag{5}$$

Let t be any positive integer. Writing $x = P_1$, (1) may be written as

$$P_1 P_t = P_{t+1} + P_{t-1}. \tag{6}$$

Now let t be any positive integer ≥ 2. It is easily checked that

$$P_2 P_t = P_{t+2} + P_t + P_{t-2}. \tag{7}$$

We shall now show that (6) and (7) are special cases of the general formula expressed by

Theorem 1: For any nonnegative integers s and t we have

$$P_s P_t = \sum_{k=0}^{s} P_{t+s-2k}.$$

We first prove Theorem 1 for the case $0 \leq s \leq t$. For $s = 0, 1, 2$, the theorem is already established. Let it hold for all $0 \leq s' < s$ and for all $t \geq s'$. Consider s, t such that $2 < s \leq t$. Using (1), we have

$$\begin{aligned} P_s P_t &= (xP_{s-1} - P_{s-2})P_t = xP_{s-1}P_t - P_{s-2}P_t \\ &= x\sum_{k=0}^{s-1} P_{s+t-1-2k} - \sum_{k=0}^{s-2} P_{s+t-2-2k} \\ &= \sum_{k=0}^{s-1} P_{s+t-2k} + \sum_{k=0}^{s-1} P_{s+t-2-2k} - \sum_{k=0}^{s-2} P_{s+t-2-2k} = \sum_{k=0}^{s} P_{s+t-2k}. \end{aligned}$$

This proves the theorem for $0 \le s \le t$.

The right-hand side of Theorem 1 appears at first sight not to be symmetric with respect to s and t. We show it to be symmetric. We first prove the simple

Lemma 1:

$$\sum_{k=1}^{n} P_{n-2k} = 0.$$

Proof: Take the terms in pairs symmetric with respect to their positions in the series. We then have

$$\sum_{k=1} P_{n-2k} = \sum_{j=1}^{[n/2]} (P_{n-2j} + P_{-(n-2j)-2}) = 0,$$

which proves the lemma.

Now let $s > t$. Put $n = s - t$ and apply Lemma 1. Then

$$\sum_{k=1}^{s-t} P_{s-t-2k} = 0. \tag{8}$$

Equality (8) together with that part of Theorem 1 already proved yield

$$\begin{aligned} P_s P_t &= \sum_{k=0}^{t} P_{s+t-2k} = \sum_{k=0}^{t} P_{s+t-2k} + \sum_{k=1}^{s-t} P_{s-t-2k} \\ &= \sum_{k=0}^{t} P_{s+t-2k} + \sum_{k=t+1}^{s} P_{s-t-2(k-t)} = \sum_{k=0}^{s} P_{s+t-2k}, \end{aligned}$$

which proves the theorem for all nonnegative integers s and t.

The following are some special cases of Theorem 1.

$$P_n^2 = \sum_{k=0}^{n} P_{2k}, \tag{9}$$

$$P_n P_{n+1} = \sum_{k=0}^{n} P_{2k+1}. \tag{10}$$

Both (9) and (10) appear in [2, p. 403].

We now have

Theorem 2: Let m and n be arbitrary integers. Then

$$P_n^2 - P_{n-m}P_{n+m} = P_{m-1}^2.$$

Proof: Case 1. $0 \le m \le n$. By using Theorem 1 for $s = n - m$, $t = n + m$, we get

$$P_{n-m}P_{n+m} = \sum_{k=0}^{n-m} P_{2(n-k)} = \sum_{k=m}^{n} P_{2k}. \tag{11}$$

Putting $m = 0$ in (11) yields (9). Subtracting (11) from (9) yields

$$P_n^2 - P_{n-m}P_{n+m} = \sum_{k=m}^{m-1} P_{2k}.$$

Using (9) again we obtain Theorem 2. This settles Case 1.

Case 2. $0 \le n < m$.

Subcase 2.1.—$n + 1 > m - 1$. Since $n + 1 \le m$, it follows that $m - 1 = n$. Then $n - m = -1$, and the theorem holds.

Subcase 2.2.—$n + 1 \le m - 1$. Then, using (4), we may write

$$\begin{aligned} P_{m-1}^2 + P_{n-m}P_{n+m} &= P_{m-1}^2 - P_{m-n-2}P_{n+m} \\ &= P_{m-1}^2 - P_{m-1-(n+1)}P_{m-1+(n+1)} = P_n^2. \end{aligned}$$

The last equality follows by applying Case 1 to $n + 1$ and $m - 1$. This completes Case 2.

The remaining cases are settled by applying similar arguments.

Corollary 1: For all integral n, we have

$$P_n^2 - P_{n-1}P_{n+1} = 1.$$

Proof: Put $m = 1$ in Theorem 2.

Writing (1) again we have $xP_n = P_{n+1} + P_{n-1}$. Then,

$$x^2P_n = x(P_{n+1} + P_{n-1}) = P_{n+2} + 2P_n + P_{n-2}.$$

By induction, it is easy to show that for all positive integers r we have

$$x^rP_n = \sum_{k=0} \binom{r}{n} P_{n+r-2k}.$$

Then,

$$\begin{aligned} P_sP_t &= \sum_{q=0} (-1)^q \binom{s-q}{q} x^{s-2q} P_t \\ &= \sum_{q=0} (-1)^q \binom{s-q}{q} \left\{ \sum_{k=0} \binom{s-2q}{k} P_{t+s-2(q+k)} \right\}. \end{aligned} \tag{12}$$

Now let $q + k = m$ be constant. Equating corresponding terms of Theorem 1 and (12) we obtain, after replacing s by n,

Theorem 3: Let m, n be nonnegative integers such that $m \le n$. Then,

$$\sum_{k=0}^{\min(m,\, n-m)} (-1)^k \binom{n-k}{k}\binom{n-2k}{m-k} = 1.$$

Corollary 2: For any nonnegative integer n, we have

$$\sum_{k=0}^{n} (-1)^k \frac{(2n-k)!}{k!((n-k)!)^2} = 1.$$

Proof: Put $n = 2m$ in Theorem 3 and replace m by n.

For nonnegative n, the polynomial $P_n(x)$ is known to be the characteristic polynomial of a simple path of length n (number of vertices in the path) [1, p. 75].

Let $C_n(x)$ be the characteristic polynomial of an n-cycle. In [6, p. 159], the following close relationship between $C_n(x)$ and $P_n(x)$ is given for $n \ge 3$:

$$C_n = C_n(x) = P_n - P_{n-2} - 2. \tag{13}$$

Using (4), we may write (13) as

$$C_n = P_n + P_{-n} - 2. \tag{14}$$

In a regular graph G, the order of regularity r is an eigenvalue of G. Therefore, we have $C_n(2) = 0$. Using (13), we obtain

$$P_n(2) = P_{n-2}(2) + 2 \tag{15}$$

for $n \geq 3$. Since $P_0(2) = 1$, $P_1(2) = 2$, $P_2(2) = 3$, it follows that for $n \geq 0$ we have

$$P_n(2) = n + 1. \tag{16}$$

This is a result in [3, 1.72].

Using (14), it is easily checked that both (15) and (16) are valid for all integral n.

Using the known expression for P_n, [6], we obtain

$$\begin{aligned} P_n &= \sum_{k=0} (-1)^k \binom{n-k}{k} x^{n-2k} = \prod_{j=1}^{n} (x - 2\cos(\pi j/(n+1))) \\ &= x^h \prod_{j=1}^{[n/2]} (x^2 - 4\cos^2(\pi j/(n+1))), \end{aligned} \tag{17}$$

where $h = n - 2[n/2]$, [8].

For positive n, (16) and (17) together imply

$$2^n \prod_{j=1}^{n} (1 - \cos(\pi j/(n+1))) = n + 1. \tag{18}$$

Taking the factors of the left-hand side in pairs, we get

Theorem 4: Let n be an integer > 1. Then,

$$\prod_{k=1}^{[n/2]} \sin(\pi k/(n+1)) = (n+1)^{\frac{1}{2}} 2^{-n/2}.$$

Theorem 4 and the left-hand side of (17) together yield

$$\prod_{k=1}^{[n/2]} \sin^2(\pi k/(n+1)) = (n+1)2^{-2} = \sum_{k=0} \left(-\frac{1}{4}\right)^k \binom{n-k}{k}. \tag{19}$$

Put $x = 0$ in (17) and let n be even and positive.

Put $n = 2m$. We then have

$$P_n(0) = (-1)^m = (-1)^m 2^n \prod_{k=1}^{m} \cos^2(\pi k/(n+1)),$$

yielding

Theorem 5: $\displaystyle\prod_{k=1}^{m} \cos(\pi k/(2m+1)) = 2^{-m}$.

Now put $x = 2i$. It then follows from (17) that

$$2^n i^n \sum_{k=0} 4^{-k} \binom{n-k}{k} = 2^n \prod_{j=1}^{n} (i - \cos(\pi j/(n+1))).$$

Again taking the factors in pairs and cancelling out, we get

(20) $$\sum_{k=0}\binom{n-k}{k}4^{-k} = \prod_{j=1}^{[n/2]}(1+\cos^2(\pi j/(n+1))).$$

By setting $x = i$ in (17), we get

$$P_n(i) = i^n\sum_{k=0}\binom{n-k}{k} = \prod_{j=1}^{n}(i - 2\cos(\pi j/(n+1))).$$

Taking the factors in pairs yields

(21) $$\sum_{k=0}\binom{n-k}{k} = \prod_{j=1}^{[n/2]}(1+4\cos^2(\pi j/(n+1))).$$

Using (3), it follows that

(22) $$P_n(i) = i^nF_{n+1}(1) = i^nf_{n+1},$$

where f_n is the nth term of the Fibonacci sequence

$$f_0 = 0,\ f_1 = 1,\ f_2 = 1,\ f_n = f_{n-1} + f_{n-2}.$$

Combining (21) and (22), we get

(23) $$f_{n+1} = \prod_{j=1}^{[n/2]}(1+4\cos^2(\pi j/(n+1))).$$

Theorem 1 and (22) together yield, for $s \le t$,

(24) $$f_sf_t = \sum_{k=0}^{s-1}(-1)^kf_{s+t-1-2k}.$$

A considerable number of identities and results on Fibonacci numbers may be derived from repeatedly using (24).

Let Y_n be the Y-graph mentioned in [6, p. 162] and let $Y_n = Y_n(x)$ be its characteristic polynomial. It follows from [6] that

(25) $$Y_n = x(P_{n-1} - P_{n-3}) = P_n - P_{n-4}.$$

We then have

$$Y_n(2) = P_n(2) - P_{n-4}(2) = 4.$$

Using the expression for Y_n in [6], we get

(26) $$Y_n = x\prod_{j=1}^{n-1}(x - 2\cos(\pi(2j-1)/2(n-1))).$$

Combining (25) and (26), we get, after setting $x = 2$,

$$2^n\prod_{j=1}^{[(n-1)/2]}(1-\cos^2(\pi(2j-1)/2(n-1))) = 4.$$

Writing n instead of $n - 1$, we get

$$\prod_{j=1}^{[n/2]}(1-\cos^2(\pi(2j-1)/2n)) = 2^{1-n},$$

and finally,

$$\prod_{j=1}^{[n/2]} \sin(\pi(2j - 1)/2n) = 2^{-\frac{1}{2}(n-1)}. \tag{27}$$

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ADDITIVE PARTITIONS OF THE POSITIVE INTEGERS

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1. INTRODUCTION

In July 1976, David L. Silverman (now deceased) discovered the following theorem.

Theorem 1: There exist sets A and B whose disjoint union is the set of positive integers so that no two distinct elements of either set have a Fibonacci number for their sum. Such a partition of the positive integers is *unique*.

Detailed studies by Alladi, Erdös, and Hoggatt [1] and, most recently, by Evans [7] further broaden the area.

The Fibonacci numbers are specified as $F_1 = 1$, $F_2 = 1$, and, for all integral n, $F_{n+2} = F_{n+1} + F_n$.

Lemma: F_{3m} is even, and F_{3m+1} and F_{3m+2} are odd.

The proof of the lemma is very straightforward.

Let us start to make such a partition into sets A and B. Now, 1 and 2 cannot be in the same set, since $1 = F_2$ and $2 = F_3$ add up to $3 = F_4$. Also, 3 and 2 cannot be in the same set, because $2 + 3 = 5 = F_5$.

$$A = \{1, 3, 6, 8, 9, 11, \ldots\};$$

$$B = \{2, 4, 5, 7, 10, 12, 13, \ldots\}.$$

If we were to proceed, we would find that there is but one choice for each integer. We also note, from $F_{n+2} = F_{n+1} + F_n$, that F_{2n} belongs in set