The former yields, upon partitioning around the sum $F_1 + F_4$, the star nonagons $\left\{ {9 \atop 4} \right\} = \left\{ {9 \atop 5} \right\}$, while the latter yields, upon partitioning around the sum $F_1 + F_2 + F_3$ or around F_3, the previous star nonagon or $\left\{ {9 \atop 2} \right\} = \left\{ {9 \atop 7} \right\}$.

I have examined all the possible star nonagons for all n inclusive of 21. When $n = 13$ and 21, this algorithm breaks down and will not produce $\left\{ {13 \atop 6} \right\}$, $\left\{ {21 \atop 4} \right\}$, and $\left\{ {21 \atop 10} \right\}$. For larger values of n, other discrepancies will appear (n need not be a Fibonacci number), but always much fewer in number than the star n-gons that are generated.

It therefore appears that the Fibonacci sequence on its own cannot exhaustively generate all star n-gons. The basic reason for this nonisomorphism is that the Fibonacci numbers are related to the combinatorics of spanning trees, the combinatorics of planar graphs, not of nonplanar graphs.

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A CONVERGENCE PROOF ABOUT AN INTEGRAL SEQUENCE

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ABSTRACT

The major theorem proven in this paper is that every positive integer necessarily converges to 1 by a finite number of iterations of the process such that, if an odd number is given, multiply by 3 and add 1; if an even number if given, divide by 2.

The first step is to show an infinite sequence generated by that iterative process is recursive. For the sake of that object, an integral variable x with $(\ell + 1)$ bits is decomposed into $(\ell + 1)$ variables $a_0, a_1, \ldots, a_\ell$, each of which is a binary variable. Then, rth iteration, starting from x, has a correspondence with a fixed polynomial of $a_0, \ldots, a_\ell$, say

$$f_r(a_0, \ldots, a_\ell),$$

no matter what value x takes. Since the number of distinct f_r's is finite in the sense of normalization, the common f_r must appear after some iterations. In the circumstances, the sequence must be recursive.

The second step is to show that a recursive segment in that sequence is (1, 2) or (2, 1). For that object, the subsequences with length 3 of that segment are classified into twelve types concerned with the middle elements modulo 12. The connectability in the segment with length 5 or larger, and the constancy of the values at the head of each segment, specify the types of subsequences, found impossible, as well as with lengths 1, 3, and 4. The only possible segment is that with length 2, like (1, 2) or (2, 1).

1. INTRODUCTION

An iterative process illustrated in Figure 1* is conjectured to necessarily converge to 1 with a finite number of iterations whenever its initial value is a positive integer. It seems, however, that no proof is yet found (see [1]).

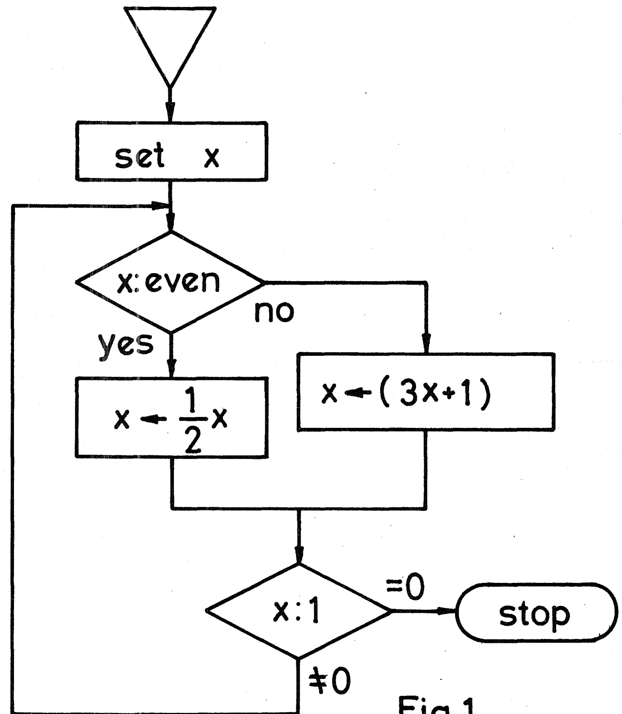

Fig.1

It is the main object to prove the truth of this conjecture.

By preliminary considerations, we easily find:

1. This iterative process is always feasible and not stopped without a reason before it attains 1.

2. If we eliminate the stopping operation after we gain 1, the sequence will be followed by a recursive sequence such as (4, 2, 1, 4, 2, 1, ...).

3. Since $(3x + 1)$ yields as an even number for odd x, then twice running on the odd side path would not occur in succession.

2. NOTATIONS AND DEFINITIONS; STATEMENT OF THE PROBLEM

ℓ is a fixed number, $\ell \in Z^+$.

a_i, $i = 0, 1, 2, \ldots, \ell$ are binary (integral) variables in the range of $[0, 1]$ and $a_0 + a_1 + \cdots + a_\ell \neq 0$.

x is a variable such that $x = 2^\ell a_0 + 2^{\ell+1} a_1 + \cdots + a_\ell$.

$\{F(a_0, a_1, \ldots, a_\ell)\}$, or $\{F(a)\}$, for short, is a set of $F(a_0, a_1, \ldots, a_\ell)$'s, the polynomials with integral coefficients about $a_0, a_1, \ldots, a_\ell$, including a polynomial with 0th order.

**AMS (MOS) Subject Classifications* (1970). Primary 10A35,10L99,68A40.

$F_\mu(a_0, a_1, \ldots, a_\ell)$, or $F_\mu(a)$, is an element of $\{F(a)\}$ composed of the the terms about the combinations of $1, a_0, \ldots, a_\ell$, as:

$$F_\mu(a) = c + c_0 a_0 + c_1 a_1 + \cdots + c_{01} a_0 a_1 + \cdots + c_{01\ldots\ell} a_0 a_1 \ldots a_\ell,$$

where $c, c_0, \ldots, c_{01\ldots\ell} \in \{0, 1\}$.

$P[F(a)]$ is a binary function about some $F(a)$ whose value is assigned as 1 or 0 according to the parity (odd or even) of the values of $F(a)$.

$AF(a)$ is the transformation of $F(a)$, where

$$AF(a) = \tfrac{1}{2}F(a) + [F(a) + \tfrac{1}{2}]P[F(a)].$$

Then, we can embed the original problem into the following:

Let ℓ be artitrarily given. Let x_0 be an arbitrary number, where

$$x_0 \in \{1, 2, 3, \ldots, 2^{\ell+1} - 1\}.$$

Then,

(i) *every x_r, $r = 1, 2, 3, \ldots$ is a positive integer, where*

$$x = \tfrac{1}{2}(3x_{r-1} + 1) \text{ for odd } x_{r-1};\ x_r = \tfrac{1}{2}x_{r-1} \text{ for even } x_{r-1},$$

and

(ii) *an infinite sequence $(x_0, x_1, \ldots)$—referred to as the S-sequence—has a recursive segment $(1, 2)$.*

3. PROCEDURES OF THE PROOF

The process of the proof is roughly classified into two stages:

1. A sequence $S:(x_0, x_1, \ldots)$ must necessarily have a subsequence with periodicity.
2. This periodical subsequence must necessarily have a recursive segment as $(1, 2)$.

4. PROOF OF PERIODICITY

Lemma 1: *Let a_i^*, $i = 0, 1, \ldots, \ell$ be some fixed numbers,*

$$a_i^* \in \{0, 1\} \text{ and } a_0^* + a_1^* + \cdots + a_\ell^* \neq 0.$$

If $a_i = a_i^*$, then $x = x_0$, where

$$x_0 = 2^\ell a_0^* + 2^{\ell-1}a^* + \cdots + a_\ell^* \text{ and } x_0 \in \{1, 2, \ldots, 2^{\ell+1} - 1\}.$$

Conversely, let x_0 be a fixed number,

$$x_0 \in \{1, 2, \ldots, 2^{\ell+1} - 1\}.$$

If $x = x_0$, then $a_i = a_i^*$, $i = 0, 1, \ldots, \ell$, where

$$a_0^* = \left[\frac{x_0}{2^\ell}\right],\ a_j^* = \left[\frac{x_0}{2^{\ell-j}}\right] - 2\left[\frac{x_0}{2^{\ell-j+1}}\right],\ j \in [1, \ell],$$

and

$$a_i^* \in \{0, 1\} \text{ with } a_0^* + \cdots + a_\ell^* \neq 0.$$

Proof: Obvious.

Corollary 1-1: $a_i^\nu = a_i^\nu$ *and* $P(a_i) = a_i$, $i = 0, 1, \ldots, \ell$, for $\forall \nu \in Z^+$.

Proof: It is obvious, since the statement does hold for arbitrary values a_i of a_i, $i = 0, 1, \ldots, \ell$.

Lemma 2: $\{F_\mu(a)\} \subseteq \{F(a)\}$.

$\{F_\mu(a)\}$ *is a finite set with 2^K elements, where $K = 2^{\ell+1}$.*

$\{F_\mu(a)\}$ is an ordered set with $\mu = 2^{K-1}c + 2^{K-2}c_0 + \cdots + c_{01\ldots\ell}$.

Proof: It is obvious, since K is the total number of the coefficients

$$c,\ c_0,\ c_1,\ \ldots,\ c_{01\ldots\ell},$$

that is,

$$K = 1 + \binom{\ell+1}{1} + \binom{\ell+1}{2} + \cdots + \binom{\ell+1}{\ell+1} = 2^{\ell+1}.$$

Corollary 2-1: There exists some $F_\mu(a)$ for each $F(a)$, $F(a) \in \{F(a)\}$, which satisfies

$$F_\mu(a) \equiv F(a) \pmod 2.$$

Proof: Obvious from the definition and Corollary 1-1.

Lemma 3: $P[F(a)] \in \{F(a)\}$.

Proof: Let $a = a^*$ be simultaneous equations $a_i = a_i^*$, $i = 0, 1, \ldots, \ell$, where each a_i^* is a fixed number with value 1 or 0.

Let $F_\mu(a) \equiv F(a) \pmod 2$ for an arbitrarily given $F(a)$ from Corollary 2-1. Then, it holds for the following congruence with fixed μ:

$$F_\mu(a^*) \equiv F(a^*) \pmod 2 \text{ for any values } a^* \in a.$$

Then, the following equalities must be satisfied:

$$\begin{aligned}1 - (-1)^{F_\mu(a^*)} &= 1 - (-1)^{F(a^*)}\\ &= [1 - (-1)][1 + (-1) + (-1)^2 + \cdots + (-1)^{F(a^*)-1}]\\ &= 2, \text{ if } F(a^*) \text{ is odd,}\\ &= 0, \text{ if } F(a^*) \text{ is even.}\end{aligned}$$

Hence,

$$2P[F(a^*)] = 1 - (-1)^{F_\mu(a^*)}.$$

Now, since $F_\mu(a)$ is congruent to such a polynomial as

$$F_\mu(a) \equiv \alpha + \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_{01} a_0 a_1 + \cdots \pmod 2,$$

where

$$\alpha,\ \alpha_0\ \ldots \in \{0, 1\} \quad\text{and}\quad \alpha \equiv c,\ \alpha_0 \equiv c_0\ \ldots \pmod 2,$$

we obtain

$$\begin{aligned}(-1)^{F_\mu(a^*)} &= (-1)^{\alpha+\alpha_0 a_0^* + \alpha_1 a_1^* + \cdots}\\ &= (-1)^\alpha (-1)^{\alpha_0 a_0^*} \cdot (-1)^{\alpha_1 a_1^*} \times \cdots\\ &= (-1)^\alpha [1 - 2P(\alpha_0 a_0^*)][1 - 2P(\alpha_1 a_1^*)] \times \cdots .\end{aligned}$$

Since

$$P(\alpha_0 a_0^*) = \alpha_0 a_0^*,\ P(\alpha_1 a_1^*) = \alpha_1 a_1^*,\ \ldots,$$

then

$$(-1)^{F_\mu(a^*)} = (-1)^\alpha (1 - 2\alpha_0 a_0^*)(1 - 2\alpha_1 a_1^*) \times \cdots,$$

so that

$$2P[F(a^*)] = 1 - (-1)^\alpha (1 - 2\alpha_0 a_0^*)(1 - 2\alpha_1 a_1^*) \times \cdots .$$

If we expand the righthand side as a polynomial of a_0^*, a_1^*, ..., then we would find that every coefficient is an even number.

Hence, we obtain the result that $P[F(a^*)]$ can be described as a fixed polynomial of a^*, a^*, ..., a_ℓ^* with integral coefficients for any values a^* of a, which is nothing but the statement of the present lemma.

Corollary 3-1:

$$P[F(a)] = P[F(a)^\nu] = \{P[F(a)]\}^\rho = \tfrac{1}{2}P\{F(a) + P[F(a)]\}, \text{ for } \forall\nu,\ \rho \in Z^+.$$

Proof: Obvious.

Lemma 4: *Let* $A^{r+1}x = A(A^r x)$, *if* $A^r x \in \{F(a)\}$, *where* $r \in \{Z^+, 0\}$. *Then*,

$$A^r x \in \{F(a)\} \text{ for every } r.$$

Proof: When $r = 0$.

Obviously, $x \in \{F(a)\}$ from the definition.

When $r \geqq 1$.

Suppose $A^{r-1}x \in \{F(a)\}$ and $P(A^{r-1}x) \in \{F(a)\}$ for some r. Let $F(a) = A^{r-1}x$ for some $F(a) \in \{F(a)\}$. Then we obtain from the definition,

$$A^r x = \tfrac{1}{2}F(a) + [F(a) + \tfrac{1}{2}]P[F(a)].$$

Since $F(a) \equiv P[F(a)] \pmod 2$ for every value of a, then we obtain

$$\tfrac{1}{2}[F(a) + P\{F(a)\}] \in \{F(a)\} \quad \text{and} \quad A^r x \in \{F(a)\}.$$

By virtue of the last lemma, we also obtain

$$P(A^r x) \in \{F(a)\}.$$

Hence, we induce that if

$$A^{r-1}x \in \{F(a)\} \quad \text{and} \quad P(A^{r-1}x) \in \{F(a)\},$$

then

$$A^r x \in \{F(a)\} \quad \text{and} \quad P(A^r x) \in \{F(a)\}.$$

Consequently, by the use of mathematical induction, we can justify the statement, since it is the truth for $r = 0$.

Lemma 5: $A^{r+1}x = \tfrac{1}{2}A^r x + (A^r x + \tfrac{1}{2})P(A^r x)$, $r = 0, 1, 2, \ldots$, *where*

$$A^{r+1}x = A(A^r x).$$

Proof: Obvious from the last lemma.

Lemma 6: *Suppose that x and a have one-to-one correspondence in the way of Lemma 1. Then, there exists a function $f_r(a) \in \{F_\mu(a)\}$ which satisfies for any values of x:*

$$A^r x \equiv f_r(a) \pmod 2,$$

where $r \in \{Z^+, 0\}$.

Proof: Lemma 4 shows that $A^r x$ yields a polynomial of $a_0, a_1, \ldots, a_\ell$ with integral coefficients.

Since $a_i^\nu = a_i$, for $\forall i$, $\forall \nu$ from Corollary 1-1, we can normalize $A^r x$ in the following way:

$$A^r x = \beta + \beta_0 a_0 + \cdots + \beta_{01\ldots\ell}a_0 a_1 \ldots a_\ell,$$

where $\beta, \beta_0 \ldots \in Z$. Here, the number of terms reduces to 2^ℓ or less.

The equation yields a congruence, modulo 2, such that $A^r x$ is congruent to a polynomial of $a_0, a_1, \ldots, a_\ell$ with coefficients 1, which is nothing but an element of the set $\{F_\mu(a)\}$.

Lemma 7: (i) *Let y be some fixed number, where $y \in \{1, 2, \ldots, (2^{\ell'} - 1)\}$, then $A^r y \not\equiv 0 \pmod 3$ for $r \geqq \ell$, where $\ell' = \ell + 1$.*

(ii) *Let $y \in Z^+$ and $y \not\equiv 0 \pmod 3$. Then, $A^r y \not\equiv 0 \pmod 3$ for $r \geqq 1$.*

(iii) *Let $y_1, y_2 \in Z^+$ and let $y_1 \not\equiv 0 \pmod 3$ and $y_2 \not\equiv 0 \pmod 3$. If $y_1 \equiv y_2 \pmod 4$, then $Ay_1 \equiv Ay_2 \not\equiv 0 \pmod 6$.*

Proof: (i) Suppose $A^r y$ is a multiple of 3 for some nonnegative integer r. Then, every $A^{r-1}y$, $A^{r-2}y$, ..., y must be an even number, because an odd number causes a number not divisible by 3 at the next step. Hence, $2^r \cdot 3 | y$. Since $y < 2^{\ell} \cdot 3$, then $A^r y$ is not a multiple of 3 for $r \geqq \ell$, contradicting the hypothesis.

(ii) Obvious, because $3 \nmid y$ does not cause $3 | Ay$.

(iii) If we construct a sequence (y_1, Ay_1, A^2y_1) or (y_2, Ay_2, A^2y_2), the sequence yields one of four types, according as the increasement or decreasement of values, illustrated as follows:

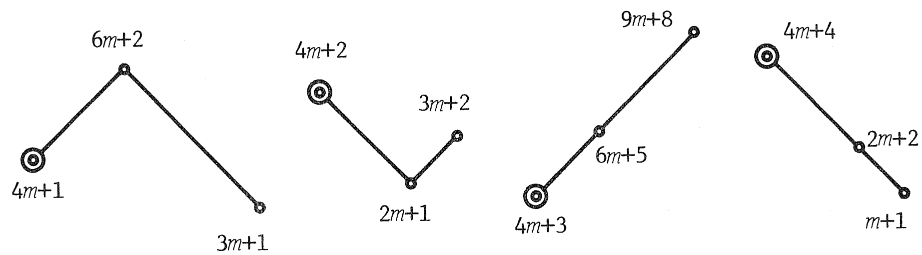

Fig.2

From the proposition, we find that (y_1, Ay_1, A^2y_1) and (y_2, Ay_2, A^2y_2) belong to a common type.

On the other hand, a sequence (y_1, Ay_1, A^2y_1) or (y_2, Ay_2, A^2y_2) can be classified about the middle element modulo 6 as follows.

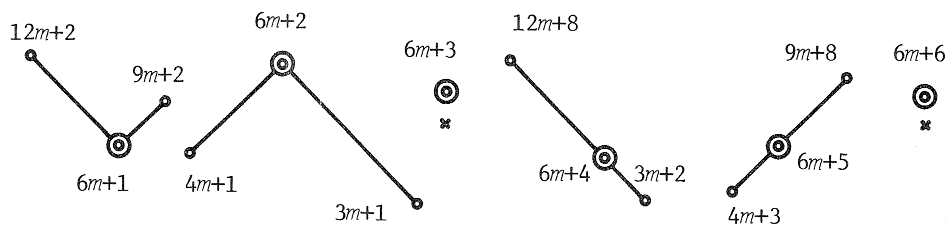

Fig.3

Since $3 \nmid y_i$ and since $3 \nmid Ay_i$ from (ii), where $i = 1, 2$, the types $(6m + 3)$ and $6m$ would not occur. Then, the types of Figure 2 have one-to-one correspondence with the types of Figure 3. Hence, the statement is justified.

Lemma 8: *Suppose that x and a have one-to-one correspondence in the way of Lemma 1. Suppose that $A^r x \equiv f_r(a)$ (mod 2) for some $f_r(a) \in \{F_\mu(a)\}$ at each $r \in \{0, 1, 2, \ldots\}$. Suppose that there exist some positive integers s and t larger than or equal to ℓ, for which $f_s(a) = f_t(a)$. Then,*

$$A^s x = A^t x \quad \textit{for every value of } x.$$

Proof: Let $x_s = A^s x|_{x=x_0}$ and $x_t = A^t x|_{x=x_0}$ for some fixed value x_0 in the range of x and let a_0^*, a_1^*, ..., a_ℓ^* or a^*, for short, have a one-to-one correspondence with x_0, as in Lemma 1. Suppose $x_s \neq x_t$, for a while, and let $x_s < x_t$. We obtain, from the propositions,

$$x_s = f_s(a^*) + (\text{an even number}),$$
$$x_t = f_t(a^*) + (\text{an odd number}),$$

which reduce to

$$x_s = f(a^*) + 2S,$$
$$x_t = f(a^*) + 2T,$$

where $f(a^*) = f_s(a^*) = f_t(a^*)$ and $S < T$. That is, $x_s \equiv x_t \pmod 2$. Moreover, since $s,\ t \geqq \ell$, we find $3 \nmid x_s$ and $3 \nmid x_t$ from the last lemma.

First, let us deal with $2A^s x$ and $2A^t x$. Since $A^s x,\ A^t x \in \{F(a)\}$, then $2A^s x,\ 2A^t x \in \{F(a)\}$; besides $P(2A^s x) = P(2A^t x) = 0$. Therefore, $A(2A^s x)$ and $A(2A^t x)$ can be defined. Since

$$A(2A^s x)|_{x=x_0} = A(2x_s) = x_s$$

as well as

$$A(2A^t x)|_{x=x} = A(2x_t) = x_t,$$

and since $2x_s \equiv 2x_t \pmod 4$ with $3 \nmid x_s x_t$, we obtain, from the last lemma,

$$A(2x_s) \equiv A(2x_t) \pmod 6, \text{ so that } x_s \equiv x_t \pmod 3.$$

That is, $S \equiv T \pmod 3$.

Now let us again deal with $A^s x$ and $A^t x$. Let $y_1 = \frac{1}{3}(2x_s + x_t)$. Then.

$$y_1 = f(a^*) + 2S + \tfrac{2}{3}(T - S).$$

Hence, y_1 is an integer with $P(y_1) = P[f(a^*)]$, and $x_s < y_1 < x_t$.

Let $y_2 = \frac{1}{3}(x_s + 2x_t)$. Then, we obtain analogously

$$y_2 = f(a^*) + 2T + \tfrac{2}{3}(S - T) \quad\text{and}\quad y_2 \in Z^+,\ P(y_2) = P[f(a^*)],\ x_s < y_2 < x_t.$$

(i) When $y_1 \not\equiv y_2 \pmod 3$:

There exists y_1 or y_2 not a multiple of 3, so that at least one of Ay_1 and Ay_2 is not divisible by 3.

(ii) When $y_1 \equiv y_2 \pmod 3$:

Then,

$$2S + \tfrac{2}{3}(T - S) \equiv 2T + \tfrac{2}{3}(S - T) \pmod 3,$$

which reduces to $T \equiv S \pmod 9$.

On the other hand, we can calculate as follows:

$$Ay_1 - Ax_s = \tfrac{1}{3}(T - S)\{1 - 2P[f(a^*)]\}.$$

Thus, $Ay_1 \equiv Ax_s \pmod 3$. Since Ax_s is not divisible by 3 for $3 \nmid x_s$, then

$$Ay_1 \not\equiv 0 \pmod 3.$$

Consequently, we can always find a number y_i, $i = 1$ or 2, which satisfies

$$\begin{cases} y_i = f(a^*) + (\text{an even number}), \\ x_s < y_i < x_t \quad\text{and} \\ Ay_i \not\equiv 0 \pmod 3. \end{cases}$$

Next let us replace a pair (x_s, x_t) with another pair (x_s, y_i) and repeat the calculations above. Then, we would obtain, analogously, some number y_i' which satisfies

$$\begin{cases} y_i' = f(a^*) + (\text{an even number}), \\ Ay_i' \not\equiv 0 \pmod 3, \text{ and} \\ x_s < y_i' < y_i. \end{cases}$$

Since this procedure can be continued infinitely, we obtain an infinite sequence of numbers y_i, y_i', y_i'', ..., which satisfies

$$x_s < \cdots < y_i'' < y_i' < y_i < x_t.$$

It is impossible in reality, because all of y_i, y_i', ..., are integers. Hence, we must conclude that $x_s = x_t$, contradicting the hypothesis in this proof.

Theorem 1: *An infinite sequence,* $S:(x_0, x_1, x_2, \ldots)$, $\forall x_0 \in Z^+$, $x_r = A^r x_0$, *is a recursive sequence.*

Proof: Lemmas 6 and 2 show that an upper limit of the number of the distinct $f_r(a)$'s is the total number of elements of the finite set $\{F_\mu(a)\}$ like 2^K. That is, the number of the distinct $f_r(a)$'s is finite.

On the other hand, since a sequence $(x_0, x_1, \ldots)$ is infinite, there exists at least one pair (x_s, x_t) which satisfies $f_s(a) = f_t(a)$ along with $s, t \geqq \ell$. For that reason, we obtain from the preceding lemmas

$$x_s = x_t.$$

Then, $x_{s+1} = x_{t+1}$, $x_{s+2} = x_{t+2}$, ..., so that the sequence is recursive. In addition, the length of a recursive segment is limited within the number of the distinct $f_r(a)$'s, like 2^K.

[NOTE: This theorem may be extended to the case of $x_0 \in Z^-$, by slight modifications.]

5. PROOF ABOUT THE LENGTH OF A PERIOD

Lemma 9: *Suppose that* $S:(x_0, x_1, \ldots, x_{s+1}, \ldots, x_t, \ldots)$, *where*

$$x_r = A^r x|_{x=x_0} \text{ for } x_0 \in \{1, 2, \ldots, 2^{\ell+1} - 1\}$$

is an infinite sequence with recursive segment $S_p:(x_{s+1}, \ldots, x_t)$.

(i) *Let* M_j, $j \in \{1, 2, 3, 4\}$ *be the total number of elements in an* S_p *with value congruent to* j *modulo* 4. *Then,* $M_1 = M_2$.

(ii) *Let* N_k, $k \in \{1, 2, \ldots, 12\}$ *be the total number of elements in an* S_p *with value congruent to* k *modulo* 12. *Then,*

$$N_1 = N_2 \quad \text{and} \quad N_3 = N_5 = N_6 = N_7 = N_8 = N_9 = N_{10} = N_{12} = 0.$$

Proof: (i) If we construct sequences U_3's$(x_r, Ax|_{x=x_r}, A^2x|_{x=x_r})$ for each element x_r of an S_p, the number of U_3's is equal to the total number of elements of an S_p, that is, $M_1 + M_2 + M_3 + M_4$. Besides, every U_3 is a subsequence of S. As we saw in Lemma 7, U_3's are classified into four-types like Figure 2. It is easily recognized that the number of each type coincides with M_1, M_2, M_3, and M_4, respectively.

On the other hand, concerning the middle elements, U_3's can be classified into six-types modulo 6 as illustrated in Figure 3. In this place, we should like to omit $6m + 3$ and $6m$, since these would not appear as a recursive element. Then, we can also recognize that the number of each type coincides with M_2, M_1, 0, M_4, M_3, and 0, respectively.

Hence, we obtain the following contrast.

M_1: total number of type $(4m + 1)$ = total number of type $(6m + 2)$,
M_2: total number of type $(4m + 2)$ = total number of type $(6m + 1)$,
M_3: total number of type $(4m + 3)$ = total number of type $(6m + 5)$,
M_4: total number of type $(4m + 4)$ = total number of type $(6m + 4)$.

Then, we can calculate the total number of the odd types in two ways: one is based on the types modulo 4 and the other is based on the types modulo 6. The result is $M_1 + M_3 = M_2 + M_3$. Hence, $M_1 = M_2$.

(ii) Let us subdivide the types of the above table modulo 12. For instance, the type $(4m+1)$ is subdivided into the types $(12m + 1)$, $(12m + 5)$, and $(12m + 9)$. Then, we can reconstruct the above table as follows:

$$M_1:\quad \left.\begin{array}{r}\text{total number of types}\\ (12m+1)\\ (12m+5)\\ (12m+9)\end{array}\right\} = \left\{\begin{array}{r}\text{total number of types}\\ (12m+2)\\ (12m+8)\\ \end{array}\right.$$

$$M_2:\quad \left.\begin{array}{r}\text{total number of types}\\ (12m+2)\\ (12m+6)\\ (12m+10)\end{array}\right\} = \left\{\begin{array}{r}\text{total number of types}\\ (12m+1)\\ (12m+7)\\ \end{array}\right.$$

$$M_3:\quad \left.\begin{array}{r}\text{total number of types}\\ (12m+3)\\ (12m+7)\\ (12m+11)\end{array}\right\} = \left\{\begin{array}{r}\text{total number of types}\\ (12m+5)\\ (12m+11)\\ \end{array}\right.$$

$$M_4:\quad \left.\begin{array}{r}\text{total number of types}\\ (12m+4)\\ (12m+8)\\ (12m+12)\end{array}\right\} = \left\{\begin{array}{r}\text{total number of types}\\ (12m+4)\\ (12m+10)\\ \end{array}\right.$$

If we omit the types with a multiple of 3 for the reason stated, and calculate in two ways, we obtain the following relations:

$$M_1 = N_1 + N_5 = N_2 + N_8,\ M_2 = N_2 + N_{10} = N_1 + N_7,$$

$$M_3 = N_{11} = N_5 + N_{11},\ M_4 = N_4 + N_8 = N_4 + N_{10}.$$

Besides, we obtain, from (i),

$$M_1 = M_2.$$

Then, they reduce to the following relations:

$$N_1 = N_2,\ N_3 = N_5 = N_6 = N_7 = N_8 = N_9 = N_{10} = N_{12} = 0.$$

Lemma 10: *Suppose that*

$$S:(x_0,\ x_1,\ \ldots),\ x_0 \in \{1,\ 2,\ \ldots,\ (2^{\ell+1} - 1)\},\ x_r = A^r x|_{x=x_0}$$

is an infinite sequence with recursive segment $S_p:(x_{s+1},\ \ldots,\ x_t)$. *Let* p *be the length of an irreducible* S_p. *Then* $p \neq 1,\ 3,\ 4$.

Proof: Since each element of S_p shows the value increasing or decreasing, according as the preceder is odd or even, then possible S_p must necessarily involve an odd element as well as an even element.

Now, let us assume, without loss of generality, that the first element of S_p is an odd number. Here $p \neq 1$, for if not, a segment would cause the value to increase. Hence, the cases to be examined are those for $p = 3$ and $p = 4$.

Let $x_{s+1} = 2R + 1$. Since $x_{s+1} \in Z^+$, then $R \in \{Z^+,\ 0\}$. Moreover, we obtain $x_{s+2} = 3R + 2$.

(i) When $p = 3$:

The cases examined are classified into four types according to the parities of x_{s+2} and x_{s+3}. Then, we can calculate x_{s+4} as a function of R.

Since $x_{s+1} = x_{s+4}$ and $R \varepsilon \{Z^+, 0\}$ must be simultaneously satisfied, we have a criterion for the existence of a recursive segment.

The result is as follows:

x_{s+2}	x_{s+3}	x_{s+4}	$x_{s+1} = x_{s+4}$	$R \varepsilon \{Z^+, 0\}$?
odd	odd	$(27R + 23)/4$	$19R + 22 = 0$	no
odd	even	$(9R + 7)/4$	$R + 3 = 0$	no
even	odd	$(9R + 8)/4$	$R + 4 = 0$	no
even	even	$(3R + 2)/4$	$5R + 2 = 0$	no

Hence, any recursive segment with length 3 does not exist.

(ii) When $p = 4$:

Analogously, we examine the simultaneous compliance of

$$x_{s+1} = x_{s+4} \neq x_{s+2} \quad \text{and} \quad R \varepsilon \{Z^+, 0\}.$$

x_{s+2}	x_{s+3}	x_{s+4}	x_{s+5}	$x_{s+1} = x_{s+5}$	$R \varepsilon \{Z^+, 0\}$?
odd	odd	odd	$(81R + 73)/8$	$R + 1 = 0$	no
odd	odd	even	$(27R + 23)/8$	$11R + 15 = 0$	no
odd	even	odd	$(27R + 25)/8$	$11R + 17 = 0$	no
even	odd	odd	$(27R + 28)/8$	$11R + 20 = 0$	no
odd	even	even	$(9R + 7)/8$	$7R + 1 = 0$	no
*even	odd	even	$(9R + 8)/8$	$R = 0$	yes
even	even	odd	$(9R + 10)/8$	$7R = 2$	no
even	even	even	$(3R + 2)/8$	$13R + 6 = 0$	no

In the above table, the asterisk marks the case of $x_{s+3} = \frac{1}{2}(3R+2)$. Since $x_{s+1} \neq x_{s+3}$ is required for an irreducible segment, then $R \neq 0$, which contradicts $x_{s+1} = x_{s+5}$.

After all, there exists no irreducible S_p with $p = 4$.

Theorem 2: *Suppose that*

$$S:(x_0, x_1, \ldots), \; x_0 \; \varepsilon \; \{1, 2, \ldots, (2^{\ell+1} - 1)\}, \; x_r = A^r x|_{x=x_0}$$

is a recursive sequence. Then, an irreducible segment of recursion, S_p *is* (1, 2) *or* (2, 1).

Proof: Since p, the length of an S_p, is not equal to 1, 3 or 4, as we saw, then the bases to be examined are limited to those of $p \geqq 5$ and $p = 2$.

(i) When $p \geqq 5$:

If we construct sequences U_4's: $(x_r, Ax|_{x=x_r}, A^2x|_{x=x_r}, A^3x|_{x=x_r})$ for each element x_r of an S_p, the number of U_4's is equal to the total number of elements of an S_p. Besides, every U_4 is a subsequence of S. As in Lemma 8, U_4's can be classified into 12-types about the second elements modulo 12 as follows.

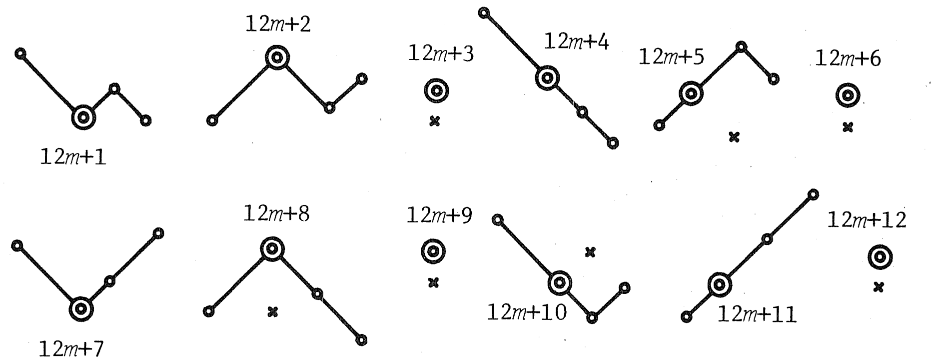

Fig.4

Since each number of type $(12m + k)$, $k \in \{1, 2, \ldots, 12\}$ coincides to the N_k stated in Lemma 8, the types $(12m + k)$, $k = 3, 5, 6, 7, 8, 9, 10, 12$ do not exist, in reality.

Now, let us construct U_5's: $(x_r, Ax|_{x=x_r}, \ldots, A^4x|_{x=x_r})$. Since every U_4 is a subsequence of S and since $p \geqq 5$, then any U_5 involves at least one combination of two U_4's such that the second, third, and fourth elements of the first U_4 overlap to the first, second, and third elements of the second U_4, respectively.

Hence, we obtain the possible combinations:

$$N_1 \to N_2;\ N_2 \to N_1;\ N_4 \to N_4;\ N_{11} \to N_{11}.$$

Since each of the latter two would not cause a recursive segment, the former two only may exist. Consequently, successive elements of S_p show the alternative increasing or decreasing of values, if it exists. In general, however, a sequence like (odd, even, odd, even, odd, ...) causes a decrease of value in the global sense, except the sequence (1, 2, 1, 2, 1, ...).

Hence, it is impossible to construct S_p with $p \geqq 5$.

(ii) When $p = 2$:

Obviously, the only S_p: (1, 2) exists, if the first element is odd.

Theorem 3: There exists an infinite sequence $(x_0, x_1, \ldots)$ generated by a recursion formula:

$$x_{r+1} = \tfrac{1}{2}(3x_r + 1), \text{ if } x_r \text{ is odd};\ x_{r+1} = \tfrac{1}{2}x_r, \text{ if } x_r \text{ is even},$$

where x_0 is arbitrarily given in Z^+.

This sequence necessarily has an element with value 1 in a finite position less than or equal to $M = 2^K + \ell$, $K = 2^{\ell+1}$ from the top of the sequence, where $K > x_0$.

Proof: Obvious from Theorems 1 and 2.

Complement: An infinite sequence $(x_0, x_1, \ldots)$ with the recursion formula like Theorem 3 starting from an arbitrary x_0 in Z^- is a recursive sequence.

Proof: Left to the reader.

6. CONCLUSION

We have proven a number-theoretical problem about a sequence, which is a computer-oriented type, but cannot be solved by any computer approach.

REFERENCE

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WEIGHTED STIRLING NUMBERS OF THE FIRST AND SECOND KIND—II

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1. INTRODUCTION

The Stirling numbers of the first and second kind can be defined by

$$(x)_n \equiv x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} S_1(n, k)x^k, \tag{1.1}$$

and

$$x^n = \sum_{k=0}^{n} S(n, k)x\cdot(x-1)\cdots(x-k+1), \tag{1.2}$$

respectively. In [6], the writer has defined *weighted* Stirling numbers of the first and second kind, $\overline{S}_1(n, k, \lambda)$ and $\overline{S}(n, k, \lambda)$, by making use of certain combinatorial properties of $S_1(n, k)$ and $S(n, k)$. Numerous properties of the generalized quantities were obtained.

The results are somewhat simpler for the related functions:

$$\begin{cases} R_1(n, k, \lambda) = \overline{S}_1(n, k+1, \lambda) + S_1(n, k) \\ R(n, k, \lambda) = \overline{S}(n, k+1, \lambda) + S(n, k). \end{cases} \tag{1.3}$$

In particular, the latter satisfy the recurrences,

$$\begin{cases} R_1(n, k, \lambda) = R_1(n, k-1, \lambda) + (n+\lambda)R_1(n, k, \lambda) \\ R(n, k, \lambda) = R(n, k-1, \lambda) + (k+\lambda)R(n, k, \lambda), \end{cases} \tag{1.4}$$

and the orthogonality relations

$$\begin{aligned} &\sum_{j=0}^{n} R(n, j, \lambda)\cdot(-1)^{j-k}R_1(j, k, \lambda) \\ &\quad = \sum_{j=0}^{n}(-1)^{n-j}R_1(n, j, \lambda)R(j, k, \lambda) = \begin{cases} 1 & (n = k) \\ 0 & (n \neq k). \end{cases} \end{aligned} \tag{1.5}$$

We have also the generating functions

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{n} R_1(n, k, \lambda)y^k = (1-x)^{-\lambda-y}, \tag{1.6}$$