Note: After completing this paper, we became aware of a similar calculation by Perry B. Wilson, in which some of the present results have been obtained (Stanford Linear Accelarator Report PEP-232, February 1977). We wish to thank Dr. S. Krinsky for calling our attention to this report.

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THE NUMBER OF PERMUTATIONS WITH A GIVEN NUMBER OF SEQUENCES

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1. Let $P(n, s)$ denote the number of permutations of $Z_n = \{1, 2, \ldots, n\}$ with s ascending or descending sequences. For example, the permutation 24315 has the ascending sequences 24, 15 and the descending sequence 431; the permutation 613254 has ascending sequences 13, 25 and descending sequences 61, 32, 54. André proved that $P(n, s)$ satisfies the recurrence

$$P(n+1, s) = sP(n, s) + 2P(n, s-1) + (n-s+1)P(n, s-2), \quad (n \geq 1), \tag{1.1}$$

where $P(0, s) = P(1, s) = \delta_{0,s}$; for proof see Netto [3, pp. 105-112].

Using (1.1), the writer [1] obtained the generating function

$$\sum_{n=0}^{\infty}(1 - x^2)^{-n/2}\frac{z^n}{n!}\sum_{s=0}^{\infty}P(n+1, s)x^{n-s} = \frac{1 - x}{1 + x}\left(\frac{\sqrt{1 - x^2} + \sin z}{x - \cos z}\right)^2. \tag{1.2}$$

However, an explicit formula for $P(n, s)$ was not found.

In the present note, we obtain an explicit result, namely

$$(1.3)\quad \begin{cases} P(2n-1, 2n-s-2) = \displaystyle\sum_{j=1}^{n}(-1)^{n-j}2^{-j+2}(2j-1)!\overline{K}_{n,j}M_{n,j,s} \\ P(2n, 2n-s-1) = \displaystyle\sum_{j=1}^{n}(-1)^{n-j}2^{-j+1}(2j)!\overline{K}_{n,j}M_{n,j,s}, \end{cases}$$

where

$$\overline{K}_{n,j} = \frac{1}{(2j)!}\sum_{t=0}^{2j}(-1)^t\binom{2j}{t}(j-t)^{2n}$$

and

$$M_{n,j,s} = \sum_{t=0}^{n-j}(-1)^t\binom{n-j}{t}\binom{n-2}{s-t}.$$

2. Put $y = \csc^2 x$. Then it is easily verified that ($D \equiv d/dx$)

$$\begin{aligned} Dy &= -2 \csc^2 x \cot x \\ D^2 y &= -4 \csc^2 x + 6 \csc^4 x \\ D^3 y &= 8 \csc^2 x \cot x - 24 \csc^4 x \cot x \\ D^4 y &= 16 \csc^2 x - 120 \csc^4 x + 120 \csc^6 x. \end{aligned}$$

Generally, we can put

(2.1) $$D^{2n-2} y = \sum_{j=1}^{n} (-1)^{n-j} a_{n,j} \csc^{2j} x \quad (n \geq 1).$$

Differentiation of (2.1) gives

$$\begin{aligned} D^{2n-1} y &= \sum_{j=1}^{n} (-1)^{n-j+1} \cdot 2j a_{n,j} \csc^{2j} x \cot x \\ D^{2n} y &= \sum_{j=1}^{n} (-1)^{n-j} a_{n,j} \{4j^2 \csc^{2j} x \cot^2 x + 2j \csc^{2j+2} x\} \\ &= \sum_{j=1}^{n} (-1)^{n-j} a_{n,j} \{2j(2j+1) \csc^{2j+2} x - 4j^2 \csc^{2j} x\}. \end{aligned}$$

Comparing this with

$$D^{2n} y = \sum_{j=1}^{n+1} (-1)^{n-j+1} a_{n+1,j} \csc^{2j} x,$$

we get the recurrence

(2.2) $$a_{n+1,j} = (2j-1)(2j-2) a_{n,j-1} + 4j^2 a_{n,j} \quad (n \geq 1).$$

It follows easily from (2.2) that $a_{n,j}$ is divisible by $(2j-1)!$. Thus, if we put

(2.3) $$a_{n,j} = (2j-1)! b_{n,j},$$

(2.2) becomes

(2.4) $$b_{n+1,j} = b_{n,j-1} + 4j^2 b_{n,j} \quad (n \geq 1).$$

Now put

(2.5) $$b_{n,j} = 2^{2n-2j} \overline{K}_{n,j},$$

so that (2.4) reduces to

(2.6) $$\overline{K}_{n+1,j} = \overline{K}_{n,j-1} + j^2 \overline{K}_{n,j} \quad (n \geq 1).$$

The $\overline{K}_{n,j}$ are evidently positive integers. Table 1 was obtained by means of (2.6).

The numbers $\overline{K}_{n,j}$ are called the divided central differences of zero [2], [5]. They are related to the $K_{n,j}$ of [2] by

(2.7) $$\overline{K}_{n,j} = K_{n+1,j}.$$

In the notation of divided central differences, we have

(2.8) $$\overline{K}_{rs} = \delta^{2s} 0^{2r} / (2s)!,$$

where

Table 1

$n \backslash j$	1	2	3	4	5
1	1				
2	1	1			
3	1	5	1		
4	1	21	30	1	
5	1	85	501	46	1

$$\delta f(x) = f\left(x + \frac{1}{2}\right) - f\left(x - \frac{1}{2}\right).$$

Thus,

$$\overline{K}_{rs} = \frac{1}{(2s)!} \sum_{t=0}^{2s} (-1)^t \binom{2s}{t} (s - t)^{2r}, \tag{2.9}$$

which is equivalent to

$$\sum_{r=1}^{\infty} \overline{K}_{r,s} \frac{x^{2r}}{(2r)!} = \frac{1}{(2s)!} (e^{(1/2)x} - e^{-(1/2)x})^s \quad (s \geq 1). \tag{2.10}$$

Substituting from (2.3) and (2.5) in (2.1), we get

$$D^{2n-2} \csc^2 x = \sum_{j=1}^{n} (-1)^{n-j} 2^{2n-2j} (2j - 1)! \overline{K}_{n,j} \csc^{2j} x \quad (n \geq 1). \tag{2.11}$$

Differentiation gives

$$D^{2n-1} \csc^2 x = -\sum_{j=1}^{n} (-1)^{n-j} 2^{2n-2j} (2j)! \overline{K}_{n,j} \csc^{2j} \phi \cot \phi \quad (n \geq 1). \tag{2.12}$$

3. Returning to the generating function (1.2), we take $x = \cos 2\phi$ and replace z by $2z$. Thus, the lefthand side becomes

$$\sum_{n=0}^{\infty} (\sin 2\phi)^{-n} \frac{2^n z^n}{n!} \sum_{s=0}^{n} P(n + 1, s) \cos^{n-s} 2\phi.$$

The right-hand side is equal to

$$\frac{1 - \cos 2\phi}{1 + \cos 2\phi} \left(\frac{\sin 2\phi + \sin 2z}{\cos 2\phi - \cos 2z}\right)^2 = \frac{\sin^2\phi}{\cos^2\phi} \left(\frac{\cos(z - \phi)}{\sin(z - \phi)}\right)^2.$$

Hence, we have

$$\sum_{n=0}^{\infty} (\sin 2\phi)^{-n} \frac{2^n z^n}{n!} \sum_{s=0}^{n} P(n + 1, s) \cos^{n-s} 2\phi = \tan^2\phi \cos^2(z - \phi).$$

Replacing ϕ by $-\phi$, this becomes

$$(3.1)\qquad \sum_{n=0}^{\infty}(-1)^n(\sin 2\phi)^{-n}\frac{2^n z^n}{n!}\sum_{=0}^{n}P(n+1,\, s)\cos^{n-s}2\phi$$
$$= \tan^2\phi\ \csc^2(z+\phi) - \tan^2\phi.$$

By Taylor's theorem,

$$\csc^2(z+\phi) = \sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{d^n}{d\phi^n}\csc^2\phi.$$

Hence, (3.1) yields

$$(-1)^n 2^n(\sin 2\phi)^{-n}\sum_{s=0}^{n}P(n+1,\, s)\cos^{n-s}2\phi = \tan^2\phi\,\frac{d^n}{d\phi^n}\csc^2\phi,$$

so that

$$(3.2)\qquad \sum_{s=0}^{n}P(n+1,\, s)\cos^{n-s}2\phi = (-1)^n\sin^{n+2}\phi\ \cos^{n-2}\phi\frac{d^n}{d\phi^n}\csc^2\phi\quad (n \geq 1).$$

Replacing n by $2n - 2$ and making use of (2.11), we get

$$(3.3)\qquad \sum_{s=0}^{2n-2}P(2n-1,\, s)\cos^{2n-s-2}2\phi$$
$$= \sin^{2n}\phi\ \cos^{2n-4}\phi\sum_{j=1}^{n}(-1)^{n-j}2^{2n-2j}(2j-1)!\overline{K}_{n,j}\csc^{2j}\phi\quad (n \geq 1).$$

Similarly, by (2.12),

$$(3.4)\qquad \sum_{s=0}^{2n-1}P(2n,\, s)\cos^{2n-s-1}2\phi$$
$$= \sin^{2n}\phi\ \cos^{2n-4}\phi\sum_{j=1}^{n}(-1)^{j}2^{2n-2j}(2j)!\overline{K}_{n,j}\csc^{2j}\phi\quad (n \geq 1).$$

We have, for $1 \leq j \leq n$,

$$2^{2n-2j}\ \sin^{2n-2j}\phi\ \cos^{2n-4}\phi = 2^{-j+2}(1-\cos 2\phi)^{n-j}(1+\cos 2\phi)^{n-2}$$
$$= 2^{-j+2}\sum_{r=0}^{n-j}\sum_{t=0}^{n-2}(-1)^r\binom{n-j}{r}\binom{n-2}{t}\cos^{r+t}2\phi.$$

For $r + t = 2n - s - 2$, comparison with (3.3) gives

$$P(2n-1,\, s) = \sum_{j=1}^{n}(-1)^{n-j}2^{-j+2}(2j-1)!\overline{K}_{n,j}\cdot\sum_{r=0}^{n-j}(-1)^r\binom{n-j}{r}\binom{n-2}{2n-r-s-2}.$$

Replacing s by $2n - s - 2$, we have

$$(3.5)\qquad P(2n-1,\, 2n-s-2) = \sum_{j=1}^{n}(-1)^{n-j}2^{-j+1}(2j-1)!\overline{K}_{n,j}$$
$$\cdot\sum_{r=0}^{n-j}(-1)^r\binom{n-j}{r}\binom{n-2}{s-r}.$$

The corresponding result for $P(2n, 2n - s - 1)$ is

$$P(2n, 2n - s - 1) = \sum_{j=1}^{n} (-1)^{n-j} 2^{-j+1} (2j)! \overline{K}_{n,j} \cdot \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r} \binom{n-2}{s-r}. \tag{3.6}$$

This completes the proof of the following theorem.

Theorem: Let $n > 1$. The number of permutations of Z_n with a given number of sequences is determined by

$$\begin{aligned} P(2n - 1, 2n - s - 2) &= \sum_{j=1}^{n} (-1)^{n-j} 2^{-j+2} (2j - 1)! \overline{K}_{n,j} M_{n,j,s} \\ P(2n, 2n - s - 1) &= \sum_{j=1}^{n} (-1)^{n-j} 2^{-j+1} (2j)! \overline{K}_{n,j} M_{n,j,s}, \end{aligned} \tag{3.7}$$

where

$$\overline{K}_{n,j} = \frac{1}{(2j)!} \sum_{t=0}^{2j} (-1)^t \binom{2j}{t} (j - t)^{2n} \tag{3.8}$$

and

$$M_{n,j,s} = \sum_{t=0}^{n-j} (-1)^t \binom{n-j}{t} \binom{n-2}{s-t}. \tag{3.9}$$

4. It follows from the definition that, for $n > 1$, $P(n, 1) = 2$. In the first of (3.7), take $s = 2n - 3$. Then, by (3.9),

$$M_{n,j,2n-3} = \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r} \binom{n-2}{2n-r-3},$$

so that $2n - r - 3 \leq n - 2$, $n - 1 \leq r$ and $j = 0$ or 1. Since $\overline{K}_{n,0} = 0$, $\overline{K}_{n,1} = 1$, $M_{n,1,2n-3} = (-1)^{n-1}$, we get

$$P(2n - 1, 1) = (-1)^{n-1} 2 \cdot (-1)^{n-1} = 2.$$

Similarly, by the second of (3.7), $P(2n, 1) = 2$.

A permutation of Z_n with $n - 1$ ascents and descents is either an up-down or a down-up permutation. Since the number of up-down permutations is equal to the number of down-up permutations, we have

$$P(n, n - 1) = 2A(n) \quad (n \geq 2), \tag{4.1}$$

where $A(n)$ is the number of up-down permutations of Z_n. Hence, in applying (3.7) to this case it is only necessary to take $s = 0$. By equation (3.9), we have $M_{n,j,0} = 0$. Thus (3.7) implies

$$\begin{aligned} A(2n - 1) &= \sum_{j=1}^{n} (-1)^{n-j} 2^{-j+1} (2j - 1)! \overline{K}_{n,j} \\ A(2n) &= \sum_{j=1}^{n} (-1)^{n-j} 2^{-j} (2j)! \overline{K}_{n,j}. \end{aligned} \tag{4.2}$$

André [3] proved that

(4.3)
$$\sum_{n=1}^{\infty} A(2n-1)\frac{x^{2n-1}}{(2n-1)!} = \tan x$$
$$\sum_{n=0}^{\infty} A(2n)\frac{x^{2n}}{(2n)!} = \sec x.$$

On the other hand, in the notation of Nörlund [4, Ch. 2],

$$\tan x = \sum_{n=1}^{\infty} (-1)^n C_{2n-1}\frac{x^{2n-1}}{(2n-1)!}$$
$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n}\frac{x^{2n}}{(2n)!},$$

where

$$C_{n-1} = 2^n(1-2^n)\frac{B_n}{2!},$$

and B_n, C_n are the Bernouli and Euler numbers, respectively. Thus, by (4.3),

(4.4)
$$A_{2n-1} = (-1)^n C_{2n-1} = (-1)^n 2^{2n}(1-2^{2n})\frac{B_{2n}}{2n}$$
$$A(2n) = (-1)^n E_{2n}.$$

Therefore, by (4.2) and (4.4),

$$2^{2n}(1-2^{2n})\frac{B_{2n}}{2n} = \sum_{j=1}^{n}(-1)^j 2^{-j+1}(2j-1)!\overline{K}_{n,j} \tag{4.5}$$

and

$$E_{2n} = \sum_{j=1}^{n}(-1)^j 2^{-j}(2j)!\overline{K}_{n,j}. \tag{4.6}$$

The representation (4.5) may be compared with the following formula in [2]:

$$(2r+1)B_{2r} = \sum_{s=1}^{r+1}(-1)^{s-1}((s-1)!)^2 s^{-1}K_{r+1,s}. \tag{4.7}$$

We remark that it is proved in [1] that

$$P(n, n-s) = \sum_{j=1}^{s} f_{sj}(n)A(n+s-j) \quad (1 \le s \le n), \tag{4.8}$$

where the $f_{sj}(n)$ are polynomials in n that satisfy $f_{s1}(n) = 1$ and

$$sf_{s+1,j}(n) = f_{s,j}(n+1) - (n-s+1)f_{s-1,j-2}(n) - 2f_{s,j-1}(n).$$

Thus, it would be of interest to evaluate the $f_{sj}(n)$.

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