A ROOT PROPERTY OF A PSI-TYPE EQUATION

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1. INTRODUCTION

By counting the number of roots between the asymptotes of the graph of

(1) $$y = f(x) = 1/x + 1/(x+1) + \cdots + 1/(x+k-1) - 1/(x+k) - \cdots - 1/(x+2k)$$

we find that $f(x)$ possesses zeros which are all negative except for one, say r, and this positive r has the interesting property that

$$[r] = k^2,$$

where the brackets denote the greatest integer function.

2. THE POSITIVE ROOT

The existence of r is obtained by direct calculation.

Theorem 1: $f(x) = 0$ possesses a positive root r, and $[r] = k^2$.

Proof:

(2) $$f(x) = \sum_{j=0}^{k-1} \frac{1}{x+j} - \sum_{j=0}^{k} \frac{1}{x+k+j} = \sum_{j=0}^{k-1} \frac{1}{(x+j)(x+k+j)} - \frac{1}{x+2k}.$$

Similarly, we remove the first term from the second summation and combine the series parts to get

(3) $$f(x) = \sum_{j=0}^{k-1} \frac{k+1}{(x+j)(x+k+1+j)} - \frac{1}{x+k}.$$

Now, if we multiply equation (2) by $x + 2k$, and equation (3) by $-(x + k)$ and add the two resulting equations, we get, after replacing x by $k^2 + h$, the result

(4) $$kf(k^2+h) = \sum_{j=0}^{k-1} \frac{1}{k^2+h+j} \cdot \frac{(1-h)k^2 - (h+j)k - h(h+j)}{(k^2+k+h+j)(k^2+k+h+1+j)}.$$

We now see at once that $f(k^2) > 0$ and $f(k^2 + 1) < 0$, since k is positive, and Theorem 1 is proved.

3. THE NUMBER OF ROOTS

The function $f(x)$ given in (1) is defined for $k = 1, 2, 3, \ldots$.

Theorem 2: $f(x) = 0$ possesses exactly $2k - 1$ negative roots and exactly one positive root.

Proof: As $x \to 0^-$, $f(x) \to -\infty$, and as $x \to -1^+$, $f(x) \to +\infty$; therefore, $f(x) = 0$ for some x in $-1 < x < 0$. Similarly for the other asymptotes, and we get

(5) $$-j + 1 < x < -j + 2,\ j = 2, 3, 4, \ldots, k,$$

implies the existence of a root in each such interval.

The branch of the curve between $-k$ and $-k + 1$ is skipped for the moment. Continuing, we find as above that (5) implies roots for

$$j = k + 2, k + 3, \ldots, 2k + 1.$$

Thus, $f(x)$ possesses at least $2k - 1$ negative roots.

Now we combine the fractions in the expression for $f(x)$ to get

$$f(x) = P(x)/[x(x+1) \ldots (x+2k)] \tag{6}$$

and observe that these negative roots are also zeros of $P(x)$, since the factors in the denominator of (6) cannot be zero at these values of x. But the degree of $P(x)$ is $2k$. Therefore, $P(x)$ possesses one more zero, and this is then the r obtained in Section 2. Q.E.D.

Remark: The branch of the curve, skipped in the above argument, then does not cut the x-axis at all.

4. THE PSI FUNCTION

The psi function, denoted by $\Psi(x)$, is defined by some authors [2, p. 241] by means of

$$\Delta^{-1}\left(\frac{1}{x}\right) = \Psi(x) + C, \tag{7}$$

where C is an arbitrary periodic function. This is the analog for defining $\ln(x)$ in the elementary calculus by means of

$$\int \frac{1}{x}\,dx = \ln(x) + c.$$

We employ (7) to obtain

$$f(x) = 2\Psi(x+k) - \Psi(x) - \Psi(x+2k+1).$$

This provides us with an iteration method for the calculation of r, starting with $r_1 = k^2$.

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RECOGNITION ALGORITHMS FOR FIBONACCI NUMBERS

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A FORTRAN, BASIC, or ALGOL program to generate Fibonacci numbers is not unfamiliar to many mathematicians. A Turing machine or a Markov algorithm to recognize Fibonacci numbers is, however, considerably more abstruse.

A Turing machine, an abstract mathematical system which can simulate many of the operations of computers, is named after A. M. Turing who first described such a machine in [2]. It consists of three main parts: (1) a finite set of states or modes; (2) a tape of infinite length with tape reader; (3) a set of instructions or rules. The tape reader can read only one character at a time,