FIBONACCI NUMBERS IN TREE COUNTS FOR MAXIMAL OUTERPLANE AND RELATED GRAPHS

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ABSTRACT

Let G denote a plane multigraph that is obtained from a maximal outerplane graph by adding a collection of multiedges. We associate with each such G an M-tree (a tree in which some vertices are designated as type M), and we observe that many such graphs can be associated with the same M-tree. Formulas for counting spanning trees are given and are used to generate some Fibonacci identities. The path P_n is shown to be the tree on n vertices whose associated graph has the maximum number of spanning trees, and a class of trees on n vertices whose associated graph yields the minimum is conjectured.

1. INTRODUCTION

The occurrence of Fibonacci numbers in spanning tree counts has been noted by many authors ([5], [6], [7], [8], [9], and [10]). In particular, the labeled *fan* on $n+2$ vertices has F_{2n+2} spanning trees, where F_n denotes the nth Fibonacci number. The fan is actually a special case of the class of maximal outerplane graphs. In [11] it is shown that any labeled maximal outerplane graph on $n+2$ vertices with exactly two vertices of degree 2 also has F_{2n+2} spanning trees. In [1] the unifying concept of the "associated tree of a maximal outerplane graph" is presented; it is shown that Fibonacci numbers occur naturally in the count of spanning trees of these graphs and depend upon the structure of the associated trees.

The purpose of this paper is to extend the idea of the associated tree to maximal outerplane graphs with multiple edges, to give formulas for counting spanning trees, and to generate Fibonacci identities. In the final section, bounds are given on the number of spanning trees of any maximal outerplane graph on $n+2$ vertices.

The associated tree T of a maximal outerplane graph G is simply the "inner dual" of G; that is, T is the graph formed by constructing the usual dual G^* and deleting the vertex in the infinite region of G. In [2] it is shown that all labeled maximal outerplane graphs that have the same associated tree have the same number of labeled spanning trees. When multiple edges are allowed in the maximal outerplane graph, the above construction can be carried out, but vertices of degree 1 or 2 in T can result from either vertices of degree 2 or 3 in G^*.

To avoid this ambiguity, we shall adopt the following convention in constructing T: place a vertex of "type R" in any interior region of G bounded by

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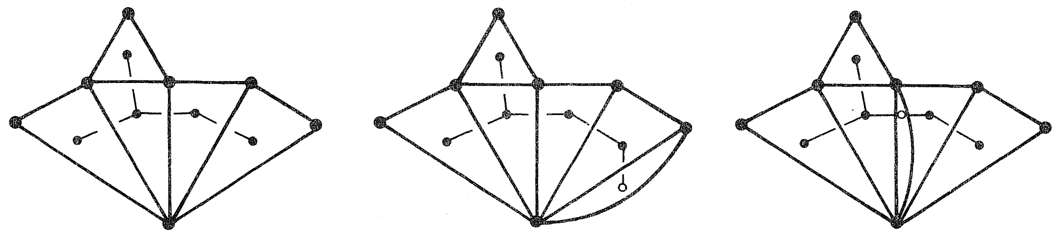
three edges and a vertex of "type M" in any interior region by a pair of multi-edges. We shall call a tree that contains any vertices of type M an *M-tree*. Figure 1 gives examples of some graphs with their associated trees and M-trees in which circles denote vertices of type M.



(a) (b) (c)

Fig. 1. Some graphs with associated trees and M-trees

If T is a tree (respectively, an M-tree), $G(T)$ will denote a maximal outerplane graph (respectively, a multigraph) associated with T. Note that there may be many nonisomorphic graphs (or multigraphs) associated with T, but that each has the same number of labeled spanning trees (STs). Consequently, we can let ST $G(T)$ denote this number. We emphasize that we are considering the edges of $G(T)$ to be labeled. For example, for the graph G_1 in Figure 2, there are 12 labeled spanning trees: four containing e_1, four containing e_2, and four that do not contain e_1 or e_2.

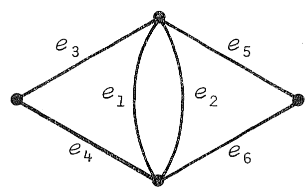


Fig. 2. A labeled multigraph G_1

It will be convenient to have a notation for some useful M-trees. As usual, P_n will denote the path with n vertices. Let $P_n^{(i)}$ denote the M-tree obtained from P_n by adjoining a path of i vertices of type M at an endpoint of P_n. A path P_n to which is attached a path of i type M vertices at one end and a path of j such vertices at the other end will be denoted $P_n^{(i,j)}$. The M-tree consisting of $P_n^{(1)}$ with P_m adjoined at the type M vertex will be denoted $P_{n,m}^{(0)}$. The M-tree constructed by adjoining a path of i vertices of type M at the $(n+1)$st vertex of P_{n+m+1} will be denoted $P_{n,m}^{(i)}$. Figure 3 shows $P_3^{(2)}$, $P_3^{(2,1)}$, $P_{2,3}^{(0)}$, and $P_{3,4}^{(2)}$.

(a) $P_3^{(2)}$ (b) $P_3^{(2,1)}$

(c) $P_{2,3}^{(0)}$ (d) $P_{3,4}^{(2)}$

Fig. 3. Some M-trees

2. TREE COUNTS AND FIBONACCI IDENTITIES

As mentioned in the previous section, it is known that

$$\mathrm{ST}\ G(P_n) = F_{2n+2}. \tag{2.1}$$

It is also known that Fibonacci numbers give the count of spanning trees of maximal outerplane graphs associated with the M-trees $P_n^{(1)}$ and $P_n^{(1,1)}$; namely,

$$\mathrm{ST}\ G(P_n^{(1)}) = F_{2n+3} \tag{2.2}$$

and

$$\mathrm{ST}\ G(P_n^{(1,1)}) = F_{2n+4}. \tag{2.3}$$

Counting the spanning trees of a graph G is often done with a basic reduction formula which sums those that contain a given edge e of G and those that do not (as in [3, p. 33]):

$$\mathrm{ST}\ G = \mathrm{ST}\ G \cdot e + \mathrm{ST}(G - \{e\}), \tag{2.4}$$

where $G \cdot e$ denotes the graph obtained by identifying the end vertices of e, and removing the self-loops. This reduction performed on $G(P_n)$ at an edge containing a vertex of degree 2 demonstrates the basic Fibonacci equation

$$F_n = F_{n-1} + F_{n-2}. \tag{2.5}$$

Repeated applications of the reduction (2.4) will give recurrences leading to several well-known identities. For example,

$$\begin{aligned}\mathrm{ST}\ G(P_n) &= \mathrm{ST}\ G(P_{n-1}) + \mathrm{ST}\ G(P_{n-1}^{(1)}) = \mathrm{ST}\ G(P_{n-2}) + \mathrm{ST}\ G(P_{n-2}^{(1)}) + \mathrm{ST}\ G(P_{n-1}^{(1)})\\ &= \mathrm{ST}\ G(P_1) + \mathrm{ST}\ G(P_1^{(1)}) + \mathrm{ST}\ G(P_2^{(1)}) + \cdots + \mathrm{ST}\ G(P_{n-1}^{(1)}),\end{aligned}$$

which, from (2.2) gives

$$F_{2n+2} = F_1 + F_3 + \cdots + F_{2n+1} \tag{2.6}$$

since $\mathrm{ST}\ G(P_1) = 3 = F_1 + F_3$.

The corresponding identity for the odd Fibonacci numbers

$$F_{2n+3} = 1 + F_2 + F_4 + \cdots + F_{2n+2} \tag{2.7}$$

follows from a similar recurrence developed by applying (2.4) to a multiple edge of $G(P_n^{(1)})$.

As another example, consider the recurrence obtained by starting again with $G(P_n)$ and alternating use of (2.4) on the resulting graphs of $G(P_k)$ and $G(P_k^{(1)})$:

$$\begin{aligned}\mathrm{ST}\ G(P_n) &= \mathrm{ST}\ G(P_{n-1}) + \mathrm{ST}\ G(P_{n-1}^{(1)}) = \mathrm{ST}\ G(P_{n-1}) + \mathrm{ST}\ G(P_{n-2}^{(1)}) + \mathrm{ST}\ G(P_{n-1})\\ &= \mathrm{ST}\ G(P_{n-1}) + \mathrm{ST}\ G(P_{n-2}^{(1)}) + \mathrm{ST}\ G(P_{n-2}) + \mathrm{ST}\ G(P_{n-2}^{(1)})\\ &= \mathrm{ST}\ G(P_{n-1}) + \mathrm{ST}\ G(P_{n-2}^{(1)}) + \mathrm{ST}\ G(P_{n-2}) + \mathrm{ST}\ G(P_{n-3}^{(1)}) + \cdots\\ &\qquad + \mathrm{ST}\ G(P_1^{(1)}) + \mathrm{ST}\ G(P_1) + \mathrm{ST}\ G(P_1^{(1)}).\end{aligned}$$

Since $\mathrm{ST}\ G(P_1^{(1)}) = F_5 = F_3 + F_2 + F_1 + 1$, this reduction yields the identity

$$F_{2n+2} = F_{2n} + F_{2n-1} + \cdots + F_2 + F_1 + 1.$$

The parallel identity

$$F_{2n+3} = F_{2n+1} + F_{2n} + \cdots + F_2 + F_1 + 1$$

can be obtained by beginning with $G(P_n^{(1)})$, and we then have the general identity

$$F_n = F_{n-2} + F_{n-3} + \cdots + F_1 + 1. \tag{2.8}$$

Other spanning tree counts that we shall find useful later can be obtained readily from (2.1), (2.2), and the reduction formula (2.4) applied at the appropriate edge:

$$\mathrm{ST}\ G(P_{h,k}^{(0)}) = F_{2h+2k+2} + F_{2h+1}F_{2k+1} \tag{2.9}$$

and
$$\text{ST } G(P_{h,k}^{(j)}) = F_{2h+2k+4} + j(F_{2h+2k+2} + F_{2h+1}F_{2k+1}). \tag{2.10}$$
Each of these counts has been discovered by other means [5].

3. FURTHER TREE COUNTING FORMULAS AND FIBONACCI IDENTITIES

A second reduction formula for counting spanning trees is useful for any 2-connected graph G with cut-set $\{u, v\}$. Let $G = H \cup K$, where $H \cap K = \{u, v\}$ and each of H and K has at least one vertex other than u or v. [Any edge (u, v) may be arbitrarily assigned either to H or to K.] Then
$$\text{ST } G = [\text{ST } H][\text{ST } K\cdot(u, v)] + [\text{ST } K][\text{ST } H\cdot(u, v)], \tag{3.1}$$
where $K\cdot(u, v)$ means graph G with vertices u and v identified. To see this, we observe that a spanning tree of G contains exactly one path between u and v. Since G is 2-connected, we may first count all the ways that this path lies entirely in H and add the ways that it lies entirely in K.

If this formula is applied to the maximal outerplane graph whose associated tree is the path P_{h+k}, we have
$$\text{ST } G(P_{h+k}) = [\text{ST } G(P_h)][\text{ST } G(P_{k-1}^{(1)})] + [\text{ST } G(P_{k-1})][\text{ST } G(P_{h-1}^{(1)})],$$
or using (2.1) and (2.2), we obtain
$$F_{2h+2k+2} = F_{2h+2}F_{2k+1} + F_{2k}F_{2h+1}, \tag{3.2}$$
which appears in [8]. A similar application on $G(P_{h+k}^{(1)})$ and use of (2.1), (2.1), and (2.3) gives
$$F_{2h+2k+3} = F_{2h+2}F_{2k+2} + F_{2h+1}F_{2k+1}. \tag{3.3}$$
Combining (3.2) and (3.3) will produce the general identity
$$F_n = F_{j+1}F_{n-j} + F_jF_{n-j-1},\ 1 \le j \le n-1. \tag{3.4}$$

The next identity will be obtained by counting the spanning trees of a graph $G(T)$ in two ways. Let S be the tree formed by joining paths of lengths j, h, and k to a vertex w (see Fig. 4). The first computation of ST $G(S)$ is obtained by applying (3.1) to $S = H \cup K$, where $H = G(P_h \cup \{w\} \cup P_k)$, and using (2.9) to get
$$\text{ST } G(S) = F_{2h+2k+4}F_{2j+1} + F_{2j}[F_{2h+2k+2} + F_{2h+1}F_{2k+1}]. \tag{3.5}$$

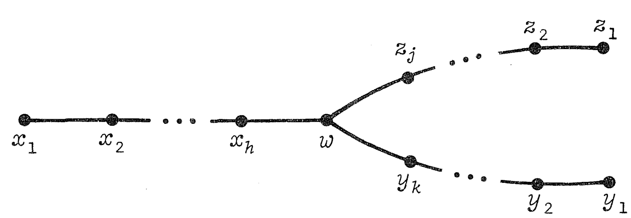


Fig. 4. The tree S

The second count is obtained by applying the reduction formula (2.4) successively to the exterior edges of $G(S)$ associated with vertices $z_j, z_{j-1}, \ldots, z_1$ and using (2.10) to get
$$\begin{aligned}\text{ST } G(S) = F_{2h+2k+4}F_{2j} &+ [F_{2h+2k+4} + 1(F_{2h+2k+2} + F_{2h+1}F_{2k+1})]F_{2j-2}\\ &+ [F_{2h+2k+4} + 2(F_{2h+2k+2} + F_{2h+1}F_{2k+1})]F_{2j-4} + \cdots\\ &+ [F_{2h+2k+4} + (j-1)(F_{2h+2k+2} + F_{2h+1}F_{2k+1})]F_2\\ &+ [F_{2h+2k+4} + j(F_{2h+2k+2} + F_{2h+1}F_{2k+1}] \cdot 1.\end{aligned}$$

Collecting terms and using identity (2.7) we have

$$\mathrm{ST}\ G(S) = F_{2h+2k+4}F_{2j+1} + \left[\sum_{r=1}^{j-1} rF_{2j-2r} + j\right]\left[F_{2h+2k+2} + F_{2h+1}F_{2k+1}\right]. \tag{3.6}$$

Equating (3.5) and (3.6) produces the identity

$$F_{2j} - j = \sum_{r=1}^{j-1} rF_{2j-2r}. \tag{3.7}$$

A corresponding formula for odd Fibonacci numbers can be obtained by starting with the multi-tree consisting of tree S with a vertex of type M attached at Z_1:

$$F_{2j+1} - 1 = \sum_{r=1}^{j-1} rF_{2j-2r+1}. \tag{3.8}$$

The identity (3.8) appears in [9], but we think that (3.7) may be new.

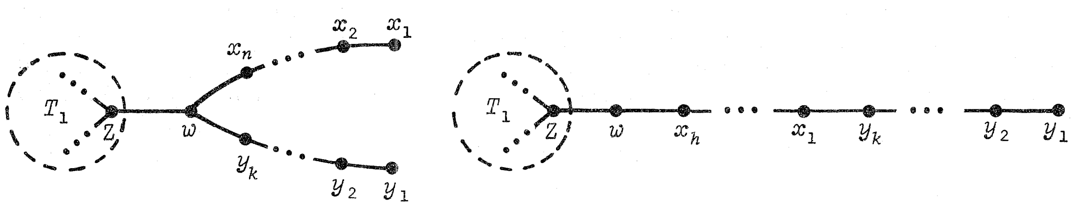
4. BOUNDS ON ST $G(T_n)$

The reduction formula (3.1) can also be used to derive a formula for "moving a branch path" in an associated tree T. Let w be a vertex of degree 3 in T with subtrees T_1, P_h, and P_k attached as in Figure 5(a). Let T' be the tree with path P_k "moved" as in Figure 5(b). Then the following formula holds

$$\mathrm{ST}\ G(T') = \mathrm{ST}\ G(T) + \mathrm{ST}[G(T_1) - (u, v)]F_{2h}F_{2k}, \tag{4.1}$$

where (u, v) is the edge in $G(T)$ separating vertices z and w. To see this, apply the reduction (3.1) to $G(T)$ and $G(T')$ with subgraphs

$$H = G(P_h \cup \{w\} \cup P_k) \quad \text{and} \quad K = G(T_1) = (u, v).$$



(a) Tree T (b) Tree T'

Fig. 5

For the graph $G(T)$ we have

$$\begin{aligned}\mathrm{ST}\ G(T) &= F_{2h+2k+4}\mathrm{ST}[G(T_1) - (u, v)] \cdot (u, v)\\ &\quad + \mathrm{ST}[G(T_1) - (u, v)][F_{2h+2k+2} + F_{2h+1}F_{2k+1}],\end{aligned}$$

where we have used (2.9). For the graph $G(T')$ we have

$$\begin{aligned}\mathrm{ST}\ G(T') &= F_{2h+2k+4}\mathrm{ST}[G(T_1) - (u, v)] \cdot (u, v)\\ &\quad + \mathrm{ST}[G(T_1) - (u, v)]F_{2h+2k+3}.\end{aligned}$$

Subtracting these equations gives us

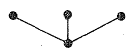
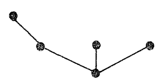
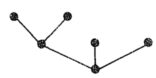
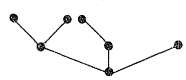
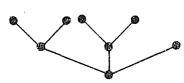
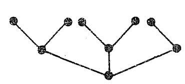
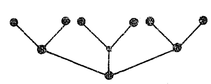
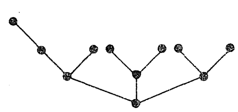
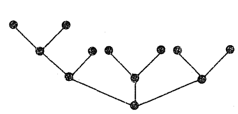
$$\begin{aligned}\mathrm{ST}\ G(T') - \mathrm{ST}\ G(T) &= \mathrm{ST}[G(T_1) - (u, v)][F_{2h+2k+1} - F_{2h+1}F_{2k+1}]\\ &= \mathrm{ST}[G(T_1) - (u, v)]F_{2h}F_{2k}\end{aligned}$$

by (3.4).

As a corollary to (4.1) we have an upper bound on ST $G(T)$: if T_n is a tree with maximum degree of a vertex equal to three, then ST $G(T_n) <$ ST $G(P_n)$.

To form a class of trees U_n whose associated maximal outerplane graph has the minimum number of spanning trees, it seems reasonable that U_n should be as "far" from P_n as possible. We conjecture that the trees U_n have the form given in Table 1.

Table 1

n	U_n	ST $G(U_n)$	ST $G(P_n) = F_{2n+2}$
1		3	3
2		8	8
3		21	21
4		54	55
5		141	144
6		360	377
7		939	987
8		2,394	2,584
9		6,237	6,765
10		15,876	17,711
11		41,391	46,368
12		105,462	121,393

The construction of the U_n can be described as follows. Let the vertex of U_1 be labeled v_1. To form U_{n+1} from U_n for $n \geq 2$, join a vertex v_{n+1} of degree 1 to U_n so that v_{n+1} is adjacent to v_i, where i is the smallest possible index subject to the requirement that all vertices of U_{n+1} have degree 3 or less.

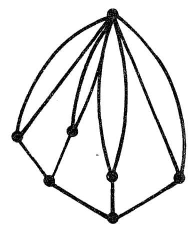
The values for ST $G(U_n)$ in Table 1 were computed using the fact that if U_n^* is the dual of $G(U_n)$, then ST $G(U_n)$ = ST U_n^*. By standard tree counting methods

$$\text{ST } U_n^* = \det(A_f A_f^t) = |a_{ij}|,$$

where A_f is the reduced incidence matrix of U_n^*. By labeling the vertices and edges of U_n^* consecutively from left to right and bottom to top, all the a_{ij} are zero except that

$$a_{ii} = 3,\ a_{12} = a_{21} = 1$$
$$a_{i,\,2i+1} = a_{2i+1,\,i} = 1,\ a_{i,\,2i+2} = a_{2i+2,\,i} = 1.$$

For example, with $n = 6$,

U_6^* is  and $\text{ST } U_6^* = \begin{vmatrix} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 \end{vmatrix}.$

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