However, Hagis also showed that, if $t^b || mn$, $t \neq p$ and $s | \sigma(t^b)$, then $\sigma(t^b)$ is divisible by a prime q satisfying (5). Hence, $s^2 | \sigma(p^a)$. Then $\sigma(p^a)$ is divisible by a prime q = 5564773, 13925333, 570421, or 985597, respectively, satisfying (5), a contradiction. Hence, $r > 21$. Q.E.D.

Lemmas 3 and 5 prove our Theorem.

REFERENCES

1. H. J. Kanold. "Untere Schranken für teilerfremde befreudete Zahlen." *Arch. Math.* 4 (1953):399-401.
2. P. Hagis, Jr. "Relatively Prime Amicable Numbers with Twenty-one Prime Divisors." *Math Mag.* 45 (1972):21-26,
3. P. Hagis, Jr. "On the Number of Prime Factors of a Pair of Relatively Prime Amicable Numbers." *Math Mag.* 48 (1975):263-266.

*****

A SPECIAL mTH-ORDER RECURRENCE RELATION

LEONARD E. FULLER
Kansas State University, Manhattan KA 66506

1. INTRODUCTION

In this paper, we consider mth-order recurrence relations whose characteristic equation has only one distinct root. We express the solution for the relation in powers of the single root. The proof for the solution depends upon a special property of factorial polynomials that is given in the first lemma. We conclude the paper by noting the simple form of the result for $m \geq 2, 3$.

2. A SPECIAL mTH-ORDER RECURRENCE RELATION

In this section, we shall consider an mth-order recurrence relation whose characteristic equation has only one distinct root λ. It is of the form

$$T = \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{m-j} \lambda^j T_{n-j}$$

with initial values $T_0, \ldots, T_{m-1}$.

Before we can prove the solution for this relation, we must establish two lemmas. The first lemma gives a useful property of the factorial polynomials. With the second lemma, we obtain an evaluation for more general polynomials. These are actually elements in the vector space $\mathbf{V}_m$ of all polynomials in j of degree less than m. This vector space has a basis that consists of the constant ${}_jP_0 = 1$ and the monic factorial polynomials in j:

$${}_jP_w = \frac{j!}{(j-w)!} = (j-0)(j-1) \ldots (j-(w-1)); \; w = 1, \ldots, m-1.$$

We will make use of the fact that the zeros of these polynomials are the integers $0, \ldots, w-1$. We are now ready to state and prove the first lemma.

Lemma 2.1: For any integers m, w where $0 \leq w < m$,

$$\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} {}_jP_w = 0.$$

We first of all observe that for $w = 0$ the factorial polynomials are just the constant 1. For the summation, we then have:

$$\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} = (1-1)^m = 0.$$

If $0 < w$ the polynomial ${}_jP_w = 0$ for $j = 0, \ldots, w - 1$. Hence in the given summation, we can omit the zero terms and start the summation at $j = w$. This gives

$$\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} {}_jP_w = \sum_{j=w}^{m} (-1)^j \binom{m}{m-j} {}_jP_w.$$

We change our summation variable to t by letting $j = w + t$. Then, we have for the summation:

$$\sum_{t=0}^{m-w} (-1)^{w+t} \binom{m}{m-w-t} {}_{w+t}P_w = (-1)^w \sum_{t=0}^{m-w} (-1)^t \frac{m!}{(m-w-t)!(w+t)!} \frac{(w+t)!}{t!}.$$

When we multiply the numerator and denominator by $(m - w)!$, we have the form:

$$(-1)^w \frac{m!}{(m-w)!} \sum_{t=0}^{m-w} (-1)^t \binom{m-w}{m-w-t} = (-1)^w \frac{m!}{(m-w)!}(1-1)^{m-w} = 0,$$

which is the result we set out to prove.

In the next lemma, we use the above result to prove a property of general polynomials $f(j) \in \mathbf{V}_m$.

<u>*Lemma 2.2*</u>: For any polynomial $f(j) \in \mathbf{V}_m$,

$$\sum_{j=1}^{m} (-1)^{j-1} \binom{m}{m-j} f(j) = f(0).$$

We shall first prove that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} f(j) = 0,$$

which is the comparable result for $f(j)$ to that of ${}_jP_w$ in Lemma 2.1.

Since $f(j) \in \mathbf{V}_m$, there exist constants c_w such that

$$f(j) = \sum_{w=0}^{m-1} {}_jP_w c_w.$$

Using this expression for $f(j)$, we have

$$\begin{aligned}\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} f(j) &= \sum_{j=0}^{m} (-1)^j \binom{m}{m-j} \sum_{w=0}^{m-1} {}_jP_w c_w \\ &= \sum_{w=0}^{m-1} c_w \sum_{j=0}^{m} (-1)^j \binom{m}{m-j} {}_jP_w = \sum_{w=0}^{m-1} c_w(0) = 0\end{aligned}$$

by Lemma 2.1.

We now break off the first term in the summation to obtain

$$\binom{m}{m} f(0) - \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{m-j} f(j) = 0,$$

so that

$$f(0) = \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{m-j} f(j),$$

which is the desired conclusion.

We shall apply this result to polynomials in $\mathbf{V}_m$ of the form

$$\binom{m+i-j}{m-u}\binom{u-1+i-j}{u-1} = \frac{(m+i-j) \cdots (i+1-j)}{(m-u)!(u-1)!(u+i-j)}.$$

The zeros of these polynomials are the integers from $i + 1$ to $m + i$ with $u + i$ omitted. We are now ready to prove our major result.

Theorem 2.3: The mth-order recurrence relation

$$T_n = \sum_{j=1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j T_{n-j}, \tag{1}$$

$T_0, \ldots, T_{m-1}$ arbitrary, has for its solution:

$$T_{m+k} = \sum_{u=1}^{m} (-1)^{u-1}\binom{m+k}{m-u}\binom{u-1+k}{u-1}\lambda^{u+k}T_{m-u}. \tag{2}$$

Before going to the proof by induction, we need to show that (2) is valid for $-m \le k < 0$. In other words, it reduces to the arbitrary value. To show this we first write (2) as a polynomial in k:

$$T_{m+k} = \sum_{u=1}^{m} (-1)^{u-1}\frac{(m+k)\ \cdots\ (1+k)}{(m-u)!(u-1)!(u+k)}\lambda^{u+k}T_{m-u}.$$

The integer k is negative, so we let $k = -s$. The polynomial in s now becomes

$$\frac{(m-s)\ \cdots\ (1-s)}{(m-u)!(u-1)!(u-s)}$$

which has for zeros the integers from 1 to m with u omitted. This means that in the summation for a fixed $k = -s$, all terms are zero except when $u = s$. The summand reduces to

$$\frac{(-1)^{s-1}(m-s)\cdots 1(-1)\cdots(1-s)}{(m-s)!(s-1)!}\lambda^0 T_{m-s} = \frac{(-1)^{s-1}(m-s)!(-1)^{s-1}(s-1)!}{(m-s)!(s-1)!}\lambda^0 T_{m-s}$$
$$= T_{m-s} = T_{m+k}.$$

This is the result we said is true.

To prove the theorem by induction on k, we first show that it is valid for $k = 0$. For this, we take $n = m$ in (1), so we have the relation

$$T_m = \sum_{j=1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j T_{m-j}.$$

For $k = 0$ in (2), we have the solution

$$T_m = \sum_{u=1}^{m} (-1)^{u-1}\binom{m}{m-u}\binom{u-1}{u-1}\lambda^u T_{m-u}.$$

These two results are equal for $u = j$.

We assume that the solution (2) is valid for $k = 0, \ldots, i-1$, and we shall show it is true for $k = i$ and, hence, for all k.

We have for (1) when $n = m + i$,

$$T_{m+i} = \sum_{j-1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j T_{m+i-j}$$

Substituting the solution (2) for T_{m+i-j}, we have

$$T_{m+i} = \sum_{j=1}^{m} (-1)^{j-1}\binom{m}{m-j}\lambda^j\left(\sum_{u=1}^{m}(-1)^{u-1}\binom{m+i-j}{m-u}\binom{u-1+i-j}{u-1}\lambda^{u+i-j}T_{m-u}\right)$$
$$= \sum_{u=1}^{m}(-1)^{u-1}\lambda^{u+i}T_{m-u}\sum_{j=1}^{m}(-1)^{j-1}\binom{m}{m-j}\binom{m+i-j}{m-u}\binom{u-1+i-j}{u-1}.$$

The inside summation involves a polynomial $f(j) \in \mathbf{V}_m$ so we can apply Lemma 2.2. Evaluating $f(0)$ gives, for the summation:

$$T_{m+i} = \sum_{u=1}^{m} (-1)^{u-1}\lambda^{u+1}T_{m-u}\binom{m+i}{m-u}\binom{u-1+i}{u-1}$$

Interchanging the order of factors in the summand gives (2) for $k = i$.

3. SPECIAL CASES

It may be helpful to consider the form of the problem for $m = 2, 3$. For $m = 2$, the relation is

$$T_n = 2\lambda T_{n-1} - \lambda^2 T_{n-2}$$

and the solution is

$$T_{2+k} = (2+k)\lambda^{1+k}T_1 - (1+k)\lambda^{2+k}T_0.$$

For $m = 3$, the relation is

$$T_n = 3\lambda T_{n-1} - 3\lambda^2 T_{n-2} + \lambda^3 T_{n-3},$$

and the solution is

$$T_{3+k} = \binom{3+k}{2}\lambda^{1+k}T_2 - \binom{3+k}{1}\binom{1+k}{1}\lambda^{2+k}T_1 + \binom{2+k}{2}\lambda^{3+k}T_0$$

$$= \frac{(3+k)(2+k)}{2}\lambda^{1+k}T_2 - \frac{(3+k)(1+k)}{1}\lambda^{2+k}T_1 + \frac{(2+k)(1+k)}{2}\lambda^{3+k}T_0$$

For other small values of m, the solutions can be written out quite readily. The form of the solution suggests a couple of other ways to write it. For instance

$$T_{2+k} = (2+k)(1+k)\left[\frac{\lambda^{1+k}}{1+k}T_1 - \frac{\lambda^{2+k}}{2+k}T_0\right] = 2\binom{2+k}{2}\left[\frac{\lambda^{1+k}}{1+k}T_1 - \frac{\lambda^{2+k}}{2+k}T_0\right]$$

and

$$T_{3+k} = \frac{(3+k)(2+k)(1+k)}{2}\left[\frac{\lambda^{1+k}}{1+k}T_2 - \frac{2\lambda^{2+k}}{2+k}T_1 + \frac{\lambda^{3+k}}{3+k}T_0\right]$$

$$= 3\binom{3+k}{3}\left[\binom{2}{2}\frac{\lambda^{1+k}}{1+k}T_2 - \binom{2}{1}\frac{\lambda^{2+k}}{2+k}T_1 + \binom{2}{0}\frac{\lambda^{3+k}}{3+k}T_0\right].$$

These two forms, when applied to the general case, give a solution of the form

$$T_{m+k} = \frac{(m+k)\ \cdots\ (1+k)}{(m-1)!}\sum_{u=1}^{m}(-1)^{u-1}\binom{m-1}{m-u}\frac{\lambda^{u+k}}{u+k}T_{m-u}$$

$$= m\binom{m+k}{m}\sum_{u=1}^{m}(-1)^{u-1}\binom{m-1}{m-u}\frac{\lambda^{u+k}}{u+k}T_{m-u}.$$

These two forms may be more suitable than the first form of the solution. Other forms could also be obtained.

*****