SOLUTIONS FOR GENERAL RECURRENCE RELATIONS

LEONARD E. FULLER
Kansas State University, Manhattan, KA 66502

1. STATEMENT OF THE PROBLEM

In a recent article [1], the author obtained representations for the solutions of certain r, s recurrence relations. In this paper we shall give representations for the solutions of general recurrence relations. In Section 4 we shall show that the results in [1] are a special case of the results of Sections 2 and 3 of this paper.

We first of all characterize all decompositions of an integer n, restricted to the first m positive integers. We define a multinomial from this that satisfies an mth-order recurrence relation with special initial conditions. Next the set of m positive integers is restricted to a subset A containing m, and a second multinomial that satisfies a recurrence relation with special initial conditions is defined.

In Section 3, we obtain solutions for comparable recurrence relations with general initial conditions. The final result gives us a solution for the general recurrence relation:

$$H_p = r_{a_1} H_{p-a_1} + \cdots + r_{a_t} H_{p-a_t};\ H_0,\ \ldots,\ H_{1-a_t} \text{ arbitrary.}$$

2. BASIC mth-ORDER RECURRENCE RELATIONS

One of the classic concepts in the theory of numbers is that of partitions of the positive integers. One of the subcases considered is for the component integers to be the set of integers from 1 to m. In this case we denote the set of all partitions of n as $P(n;m)$. The number of elements in this set is $P_m(n)$. A given partition can be characterized by a set of integers k_i. That is,

$$n = 1k_1 + \cdots + mk_m.$$

The integers k_i are referred to as the frequency of i in the given partitions. We refer to this given partition as $p(k, n; m)$.

For a given $p(k, n; m)$, we can represent n as a sum of integers from 1 to m in

$$\frac{(k_1 + \cdots + k_m)!}{k_1! \ \ldots \ k_m!}$$

ways. Each such representation is called a "decomposition of n" (some authors call them "compositions"). We denote this expression as $d_m(k, n)$. It is the number of decompositions of the partition $p(k, n; m)$.

This expression has a property that we shall find useful:

$$\frac{(k_1 + \cdots + k_m)!}{k_1! \ \ldots \ k_m!} = \frac{(k_1 + \cdots + k_m - 1)!}{k_1! \ \ldots \ k_m!} \sum_{s=1}^{m} k_s$$

$$= \sum_{s=1}^{m} \frac{(k_1 + \cdots + k_m - 1)!}{k_1! \ \ldots \ (k_s - 1)! \ \ldots \ k_m!}. \tag{2.1}$$

Symbolically we have

$$d_m(k, n) = \sum_{s=1}^{m} d_m(k(s),\ n - s),$$

where $d_m(k(s), n - s) = 0$ if $k_s = 0$. Otherwise, it is the number of decompositions for the partition of $n - s$ where all the k_i are the same as for the k partition of n except that k_s is reduced by 1.

We use this number of decompositions to define a multinomial. We then show that it is the solution for a special recurrence relation. Let

$$U_n = \sum_{P(n;m)} d_m(k, n) r_1^{k_1} \ldots r_m^{k_m},$$

that is, we sum over all partitions of n, a multinomial in $r_1, \ldots, r_m$ whose coefficients are the number of decompositions of the given partition. We can now prove our first theorem.

Theorem 2.1: The multinomial U_n satisfies the recurrence relation

$$U_t = \sum_{s=1}^{m} r_s U_{t-s};\ U_0 = 1,\ U_{-1} = \cdots = U_{1-m} = 0.$$

By applying property (1) to the definition of U_n, we have

$$\begin{aligned} U_n &= \sum_{P(n;m)} d_m(k, n) r_1^{k_1} \ldots r_m^{k_m} \\ &= \sum_{P(n;m)} \sum_{s=1}^{m} d_m(k(s), n - s) r_1^{k_1} \ldots r_m^{k_m} \\ &= \sum_{s=1}^{m} r_s \sum_{P(n-s;m)} d_m(k(s), n - s) r_1^{k_1} \ldots r_s^{k_s - 1} \ldots r_m^{k_m} \\ &= \sum_{s=1}^{m} r_s U_{n-s}. \end{aligned}$$

We have used the fact that decreasing the frequency of s by 1 gives the restricted partitions of $n - s$. If s has a frequency of 0 for a given partition, then the corresponding term in the summation on s is 0.

For $n < m$, the frequencies for the integers $n + 1$ to m would all be zero. Hence the summation can be terminated at n. However, if we choose $U_{-1} = \ldots = U_{1-m} = 0$, then we do not need any restriction. This gives $m - 1$ initial conditions. For the mth one, we shall choose $U_0 = 1$. This is logical, since all factorials are 0! and all exponents of the r_i are 0. This would give a value of 1. Hence the U_n does satisfy the prescribed recurrence relation.

What we have just proved for the case of the restricted partitions of n can be specialized for a proper subset $A = \{a_1, \ldots, a_j\}$ of the integers from 1 to m. For convenience, we assume m is in A. The set of all partitions of n restricted to the set A we label $P(n; A)$. The number of elements in this set is $P_A(n)$. A given partition can be characterized by a set of frequencies k_i, so that

$$n = a_1 k_{a_1} + \cdots + a_j k_{a_j}.$$

We refer to this given partition as $p(k, n; a)$.

For each such partition, we can represent n as a sum of integers in A in

$$\left(\sum_{i=1}^{j} k_{a_i}\right)! \Big/ \prod_{i=1}^{i} k_{a_i}!$$

ways. We denote this number as $d_A(k, n)$, that is, there are this many decompositions of the given partition, restricted to A. We can define the following multinomial

$$V_n = \sum_{P(n;A)} d_A(k, n) \prod_{q \in A} r_q^{k_q}.$$

We then have the following theorem.

Theorem 2.2: The multinomial V_n satisfies the recurrence relation

$$V_t = \sum_{s \in A} r_s V_{t-s}; \; V_0 = 1, \; V_{-1} = \cdots = V_{1-m} = 0.$$

This theorem is a special case of Theorem 2.1. First of all, the restriction to the set A means that the frequencies $k_i = 0$ if $i \varepsilon A$. This means that for each partition of n there is no s corresponding to each such i in the solution. Hence s is summed only on A. Furthermore, since the corresponding r_i is always to the zero power, we drop these r_i in the multinomial. The number of initial conditions is dependent only on the largest integer in A, which is assumed to be m.

3. GENERAL RECURRENCE RELATIONS

Using the results of the last section, we can obtain solutions for recurrence relations with arbitrary initial conditions. We shall consider two cases that are comparable to those in the last section. Our solutions will involve the U_n and V_n, respectively.

Theorem 3.1: The solution for the recurrence relation

$$G_t = \sum_{s=1}^{m} r_s G_{t-s}; \; G_0, \; \ldots, \; G_{1-m} \text{ arbitrary}, \tag{3.1}$$

is given by

$$G_n = \sum_{j=1}^{m} \sum_{q=j}^{m} r_q U_{n-j} G_{j-q}. \tag{3.2}$$

For $n = 1$ in (3.2) the $U_{n-j} = U_{i-j}$ is zero except for $j = 1$. In this case $U_0 = 1$. The double summation reduces to

$$G_1 = \sum_{q=1}^{m} r_q G_{1-q},$$

which is (3.1) for $t = 1$ and $q = 2$.

For $n = 2$ in (3.2) the $U_{2-j} = 0$ for $j > 2$. We then have

$$G_2 = U_1 \sum_{q=1}^{m} r_q G_{1-q} + U_0 \sum_{q=2}^{m} r_q G_{2-q}.$$

From the previous section, we have that $U_0 = 1$ and $U_1 = r_1$. Also, by (3.1) the first sum is G_1. Hence we have

$$G_2 = r_1 G_1 + \sum_{q=2}^{m} r_q G_{2-q} = \sum_{q=1}^{m} r_q G_{2-q},$$

which is (3.1) for $t = 2$ and $s = q$.

We assume that (3.2) is a valid solution for $n = 1, \ldots, i - 1$. For $t = i$ in (3.1),

$$G_i = \sum_{s=1}^{m} r_s G_{i-s}.$$

We have assumed solutions for all the G_{i-s} in this summation. Hence on substitution into this expression, we obtain

$$\begin{aligned} G_i &= \sum_{s=1}^{m} r_s \sum_{j=1}^{m} \sum_{q=j}^{m} r_q U_{i-s-j} G_{j-q} \\ &= \sum_{j=1}^{m} \sum_{q=j}^{m} r_q \left(\sum_{s=1}^{m} r_s U_{i-s-j} \right) G_{j-q} \\ &= \sum_{j=1}^{m} \sum_{q=j}^{m} r_q U_{i-j} G_{j-q}. \end{aligned}$$

At the last step we use the fact that U_n satisfies a recurrence relation. This final result is (3.2) for $n = i$.

We are now ready to present the solution to a general recurrence relation. We assume that set A has the properties of the last section.

Theorem 3.2: The solution for the recurrence relation

$$H_t = \sum_{s \in A} r_s H_{t-s};\ H_0,\ \ldots,\ H_{1-m} \text{ arbitrary}, \tag{3.3}$$

is given by

$$H_n = \sum_{q \in A} \sum_{j=1}^{q} r_q V_{n-j} H_{j-q}. \tag{3.4}$$

This theorem follows from Theorem 3.1, just as Theorem 2.2 followed from Theorem 2.1. For convenience, we have interchanged the order of summations in the solution so that it is easier to adapt to the restriction on q.

4. SOME SPECIAL CASES

In this section we shall consider some special cases of the results of Sections 2 and 3. They are for both the U_n and G_n relations for $m = 2$.

The restricted partitions of n for $m = 2$ would be of the form $n = k_1 + 2k_2$. The summation over all such partitions can be represented by a summation on j when $j = k_2$. Then $k_i = n - 2j$, and the summation is from 0 to $[n/2]$. The number of decompositions for a given partition would be given by

$$d_2(k,\ n) = \frac{(n - 2j + j)!}{(n - 2j)!\,j!} = \binom{n - j}{j}.$$

The solution for U_n in this case is

$$U_n = \sum_{j=0}^{[\frac{n}{2}]} \binom{n - j}{j} r_1^{n-2j} r_2^{j}.$$

For the more general G_n relation we have

$$\begin{aligned} G_n &= \sum_{j=1}^{2} \sum_{q=j}^{2} r_q U_{n-j} G_{j-q} = (r_1 U_{n-1} G_0 + r_2 U_{n-1} G_{-1}) + (r_2 U_{n-2} G_0) \\ &= (r_1 U_{n-1} + r_2 U_{n-2}) G_0 + r_2 U_{n-1} G_{-1} = U_n G_0 + r_2 U_{n-1} G_{-1}. \end{aligned}$$

Substituting in the solution for U_n and U_{n-1},

$$G_n = \sum_{j=0}^{[\frac{n}{2}]} \binom{n - j}{j} r_1^{n-2j} r_2^{j} G_0 + \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n - 1 - j}{j} r_1^{n-1-2j} r_2^{j+1} G_{-1}.$$

We change the second index of summation by replacing $j + 1$ by j, as follows:

$$G_n = \sum_{j=0}^{[\frac{n}{2}]} \binom{n-j}{j} r_1^{n-2j} r_2^j G_0 + \sum_{j=1}^{[\frac{n+1}{2}]} \binom{n-j}{j-1} r_1^{n+1-2j} r_2^j G_{-1}.$$

The author gave representations for some special recurrence relations in a previous paper [1]. We shall now show that these were particular cases of the U_n and G_n relations for $m = 2$.

The first relation presented was a generalized Fibonacci sequence,

$$G_k = rG_{k-1} + sG_{k-2};\ G_0 = 0,\ G_1 = 1,$$

which has the solution

$$G_k = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k-1-j}{j} r^{k-1-2j} s^j.$$

We observe that both our indexing and the constants of the relations are different. To reconcile them, we replace n by $k - 1$, r_1 by r, and r_2 by s in the U_n solution. This gives us the desired result.

As a special case, when $r = s = 1$ we have the Fibonacci sequence. The general term would be given by

$$F_k = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k-1-j}{j},$$

which is the number of decompositions of $k - 1$ restricted to 1 and 2.

Another sequence presented in [1] is the generalized Lucas sequence M_k, for which

$$M_k = rM_{k-1} + sM_{k-2};\ M_0 = 2,\ M_1 = r.$$

To obtain the solution we specialize the G_n for $m = 2$. We replace n by $k - 1$, r_1 by r, r_2 by s, G_0 by r, and G_{-1} by 2. We have

$$M = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k-1-j}{j} r^{k-1-2j} s^j r + \sum_{j=1}^{[\frac{k-1}{2}]} \binom{k-1-j}{j-1} r^{k-2j} s^j 2.$$

We observe that the powers of r and s in both sums are the same. Hence we combine them into a single sum. It can be verified that this yields

$$M_k = \sum_{j=0}^{[\frac{k}{2}]} \frac{k}{k-j} \binom{k-j}{j} r^{k-2j} s^j,$$

which is the solution given in [1].

The third relation discussed in [1] is

$$U_k = rU_{k-1} + sU_{k-2};\ U_1,\ U_0 \text{ arbitrary.}$$

We can identify this with our G_n relation if we let $n = k-1$, $r_1 = r$, $r_2 = s$, $G_0 = U_1$, and $G_{-1} = U_0$. This gives

$$U_k = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k-1-j}{j} r^{k-1-2j} s^j U_1 + \sum_{j=1}^{[\frac{k}{2}]} \binom{k-1-j}{j-1} r^{k-2j} s^j U_0.$$

Applying some algebra to combine the two sum yields the following solution:

$$U_k = \sum_{j=0}^{[\frac{k}{2}]} \binom{k-j}{j} \frac{(k-2j)U_1 + jrU_0}{k-j} r^{k-1-2j} s^j.$$

This can also be verified directly.

In a future paper we shall show that there are generating functions for the four recurrence relations given in this paper. These can also be used for the special cases of this section. We can use them to generate with a computer as many terms in a given recurrence relation as desired.

REFERENCE

1. L. E. Fuller. "Representations for r, s Recurrence Relations." *The Fibonacci Quarterly* 18 (1980):129-135.

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ON GENERATING FUNCTIONS AND DOUBLE SERIES EXPANSIONS

M. E. COHEN and H. S. SUN
California State University, Fresno, CA 93740

1. INTRODUCTION

Recently, Weiss *et al.* [9] gave a direct proof of a result due to Narayana [8] and Kreweras [6]:

$$\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} u^r v^s = \frac{1}{2}[1-u-v-(1-2(u+v)+(u-v)^2)^{1/2}]. \tag{1.1}$$

A special case of Theorem 1a of this pàper is a five-parameter generalization of (1.1):

$$\sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{u^k v^p}{(\alpha+1+gk+hp)} \binom{\alpha+gk+k+hp}{k} \binom{\beta+gck+hcp+p}{p}$$
$$= \frac{(1+z)^{\alpha+1}(1+y)^{\beta+1}}{(\alpha+1)} {}_2F_1\left[\begin{matrix} 1, 1+\beta-c-\alpha c, \\ (\alpha+1+h)/h, \end{matrix} \; -y\right], \tag{1.2}$$

where

$$u = \frac{z}{(1+z)^{g+1}(1+y)^{gc}}, \quad v = \frac{y}{(1+z)^h(1+y)^{hc+1}}.$$

See Luke [7, Sec. 6.10] for a discussion of Padé approximation for the hypergeometric function on the right-hand side of (1.2). Letting

$$g=-1,\ h=-1,\ c=1,\ \alpha=-2, \text{ and } \beta=-2$$

in (1.2) and some manipulation will give (1.1).

Equation (1.2) also appears to be an extension of the important equation (6.1) of Gould [5], to which it reduces for $z = 0$.

An interesting simplification of (1.2) is the case $\beta = \alpha c + c - 1$, giving:

$$\sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{u^k v^p}{(\alpha+1+gk+hp)} \binom{\alpha+gk+k+hp}{k} \binom{\alpha c+c-1+gck+hcp+p}{p}$$
$$= \frac{(1+z)^{\alpha+1}(1+y)^{\alpha c+c}}{(1+\alpha)}. \tag{1.3}$$