of 1's. At the end of a loop, the α's are changed to deltas and more 1's are changed into α's to correspond to the number of β's which begin the string. The deltas are then changed to β's. Thus, after one loop, the number of α's has changed from F_i to F_{i+1}, and the number of β's has changed from F_{i+1} to

$$F_i + F_{i+1} = F_{i+2}.$$

If there are no more 1's to be changed at the end of a loop, the Markov algorithm stops at rule 12, indicating that the original string of 1's was a Fibonacci number. If, however, the string was not a Fibonacci number, the Markov algorithm jumps out of the loop in midstream of changing 1's to α's and goes into an endless loop at rule 14 after changing the α's back to 1's.

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ON SOME CONJECTURES OF GOULD ON THE PARITIES OF THE BINOMIAL COEFFICIENTS

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In studying the parities of the binomial coefficients, Gould [1] noted several interesting relationships about the signs of the sequence of numbers

$$(-1)^{\binom{n}{0}},\ (-1)^{\binom{n}{1}},\ \ldots,\ (-1)^{\binom{n}{n}}.$$

Further interesting relationships may be discovered by converting each such sequence to a binary number, $f(2,\ n)$, by

$$f(x,\ n) = \sum_{k=0}^{n} x^k \frac{1 - (-1)^{\binom{n}{k}}}{2} \tag{1}$$

and then comparing the numbers of the sequence $f(2,\ 0)$, $f(2,\ 1)$, $f(2,\ 2)$, ... The following conjectures were then proposed by Gould.

Conjecture 1: $f(2,\ 2^m - 1) = 2^{2^m} - 1$.

Conjecture 2: $f(2,\ 2\ ) = 2^{2^m} + 1$.

Conjecture 3: $f(x,\ 2n + 1) = (x + 1)f(x,\ 2n)$.

We will prove these conjectures and present some related results.

The following lemma provides a convenient recursive scheme for generating the sequence of numbers $f(x,\ 0)$, $f(x,\ 1)$, ... We use the notation $(.)_x$ to denote the representation of a number to the base x.

Lemma 1: The sequence $f(x,\ n)$ may be defined by $f(x,\ 0) = 1$, and if

$$f(x,\ n - 1) = (a_{n-1},\ \ldots,\ a_0)_x$$

for $n > 0$, then

$$f(x, n) = x^n + 1 + \sum_{k=1}^{n-1} x^k |a_k - a_{k-1}|. \tag{2}$$

Proof: It follows directly from (1) that

$$f(x, n) = x^n + 1 + \sum_{k=1}^{n-1} x^k \frac{1 - (-1)^{\binom{n}{k}}}{2}.$$

By the well-known recursion for binomial coefficients,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

so that

$$(-1)^{\binom{n}{k}} = \begin{cases} +1 & \text{if } (-1)^{\binom{n-1}{k}} = (-1)^{\binom{n-1}{k-1}} \\ -1 & \text{otherwise.} \end{cases}$$

Therefore,

$$\frac{1 - (-1)^{\binom{n}{k}}}{2} = |a_k - a_{k-1}| \quad \text{for } n - 1 \geq k \geq 1.$$

Theorem 1: $f(x, 2^m - 1) = \sum_{k=0}^{2^m - 1} x^k.$

Proof: The theorem is clearly satisfied for $m = 1$. Assume that

$$f(x, 2^m - 1) = \sum_{k=0}^{2^m - 1} x^k = (a_{2^m - 1}, \ldots, a_0)_x,$$

where $a_k = 1$ for $2^m - 1 \geq k \geq 0$. By Lemma 1,

$$f(x, 2^m) = x^{2^m} f(x, 0) + f(x, 0).$$

We may apply (2) to both parts of $f(x, 2^m)$ independently for $2^m - 1$ times, and then add the results to obtain

$$f(x, 2^m + 2^m - 1) = x^{2^m} f(x, 2^m - 1) + f(x, 2^m - 1).$$

By the induction hypothesis,

$$f(x, 2^{m+1} - 1) = x^{2^m} \sum_{k=0}^{2^m - 1} x^k + \sum_{k=0}^{2^m - 1} x^k = \sum_{k=2^m}^{2^{(m+1)} - 1} x^k + \sum_{k=0}^{2^m - 1} x^k = \sum_{k=0}^{2^{(m+1)} - 1} x^k.$$

Corollary 1 (Conjecture 1): $f(2, 2^m - 1) = 2^{2^m} - 1.$

Corollary 2: $f(x, 2^m) = x^{2^m} + 1.$

Proof: Apply (2) to the result of Theorem 1.

Corollary 3 (Conjecture 2): $f(2, 2^m) = 2^{2^m} + 1.$

Let $L(n)$ denote $2^{\lfloor \log_2 n \rfloor}$, where $\lfloor y \rfloor$ denotes the integer part of y. Examining each number $f(x, n)$ as a number to the base x, the following striking symmetry may be noticed: the sequence of the least significant $L(n)$ digits of $f(x, n)$, is equal to the sequence of the next most significant $L(n)$ digits of $f(x, n)$, which is also equal to the sequence of the least most significant $L(n)$ digits of $f(x, n - L(n))$. The following lemma, which is based on this symmetry provides another recursive scheme for generating the sequence $f(x, 0), f(x, 1), \ldots$.

Lemma 2: For $n > 0$, $f(x, n) \bmod (x^{L(n)}) = \left\lfloor \frac{f(x, n)}{x^{L(n)}} \right\rfloor = f(x, n - L(n)).$

Proof: We distinguish between the two cases of whether or not there exists an integer m such that $n = 2^m$. If $n = 2^m$ for some integer m, then from Corollary 2 it follows that $f(x, n) = x^n + 1$ and

$$f(x, n) \bmod (x^n) = 1 = \left\lfloor \frac{f(x, n)}{x^n} \right\rfloor.$$

Furthermore, since $L(n) = n$, it follows that $f(x, n - L(n)) = f(x, 0) = 1$, and the lemma is established for this case.

For the case $n \neq L(n)$, it follows from Corollary 2 that

$$f(x, L(n)) = x^{L(n)} f(x, 0) + f(x, 0).$$

Applying (2) to $f(x, L(n))$ for $n - L(n)$ times, we may treat the two parts independently and

$$f(x, n) = x^{L(n)} f(x, n - L(n)) + f(x, n - L(n)).$$

Consequently,

$$f(x, n) \bmod (x^{L(n)}) = \left\lfloor \frac{f(x, n)}{x^{L(n)}} \right\rfloor = f(x, n - L(n)).$$

We are now in a position to prove Conjecture 3.

Theorem 2 (Conjecture 3): $f(x, 2n + 1) = (x + 1)f(x, 2n)$.

Proof: Since $x + 1 = (1, 1)_x$, the theorem will follow from elementary rules of multiplication in the base x if we can prove that when $f(x, 2n)$ is expressed in the base x, no pair of consecutive digits are 1's. We will prove this property by induction. This is certainly true for $f(x, 0) = (1)_x$. For arbitrary $n > 0$, let

$$f(x, 2n) = (a_{2L(2n)-1}, \ldots, a_0)_x.$$

By Lemma 2, each half of this number is equal to $f(x, 2n - L(2n))$ which, by the induction hypothesis, does not have two consecutive 1's when expressed in the base x. It remains to be shown that $a_{L(2n)-1} = 0$. But, by Lemma 2,

$$a_{L(2n)-1} = a_{2L(2n)-1},$$

and $a_{2L(2n)-1}$ cannot be equal to 1 because $f(x, 2n) < x^{2L(2n)-1}$.

We conclude with a final observation on the sequence of numbers $f(x, n)$. Examining the $2^m \times 2^m$ binary matrix in which the entry a_{ij} is the jth digit of $f(x, i - 1)$, we note that the matrix is symmetric about its major diagonal.

REFERENCE

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