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# COMPLEX FIBONACCI NUMBERS


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## 1. INTRODUCTION

In this note, a new approach is taken toward the significant extension of Fibonacci numbers into the complex plane. Two differing methods for defining such numbers have been considered previously by Horadam [4] and Berzsenyi [2]. It will be seen that the new numbers include Horadam's as a special case, and that they have a symmetry condition which is not satisfied by the numbers considered by Berzsenyi.

The latter defined a set of complex numbers at the Gaussian integers, such that the characteristic Fibonacci recurrence relation is satisfied at any horizontal triple of adjacent points. The numbers to be defined here will have the symmetric condition that the Fibonacci recurrence occurs on any horizontal or vertical triple of adjacent points.

Certain recurrence equations satisfied by the new numbers are outlined, and using them, some interesting new Fibonacci identities are readily obtained. Finally, it is shown that the numbers generalize in a natural manner to higher dimensions.

## 2. THE COMPLEX FIBONACCI NUMBERS

The numbers, to be denoted by $G(n, m)$, will be defined at the set of Gaussian integers $(n, m) = n + im$, where $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. By direct analogy with the classical Fibonacci recurrence

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1, \tag{2.1}$$

the numbers $G(n, m)$ will be required to satisfy the following two-dimensional recurrence

$$G(n + 2, m) = G(n + 1, m) + G(n, m), \tag{2.2}$$

$$G(n, m + 2) = G(n, m + 1) + G(n, m), \tag{2.3}$$

where $$G(0, 0) = 0,\ G(1, 0) = 1,\ G(0, 1) = i,\ G(1, 1) = 1 + i. \tag{2.4}$$

The conditions (2.2), (2.3), and (2.4) are sufficient to specify the unique value of $G(n, m)$ at each point $(n, m)$ in the plane, and the actual value of $G(n, m)$ will now be obtained.

From (2.2), the case $m = 0$ gives

$$G(n + 2, 0) = G(n + 1, 0) + G(n, 0);\ G(0, 0) = 0,\ G(1, 0) = 1$$

and hence that

$$G(n, 0) = F_n, \tag{2.5}$$

the classical Fibonacci sequence.

The case $m = 1$ gives the recurrence

$$G(n + 2, 1) = G(n + 1, 1) + G(n, 1); \; G(0, 1) = i, \; G(1, 1) = 1 + i,$$

which is an example of the well-known generalized Fibonacci sequence considered by Horadam [3] that satisfies

$$G(n, 1) = F_{n-1}G(0, 1) + F_n G(1, 1).$$

By substitution

$$G(n, 1) = iF_{n-1} + (1 + i)F_n = F_n + i(F_{n-1} + F_n),$$

and so by (2.1)

$$G(n, 1) = F_n + iF_{n+1}. \tag{2.6}$$

Recurrence (2.3) together with initial values (2.5) and (2.6) specify another generalized Fibonacci sequence, so that

$$\begin{aligned} G(n, m) &= F_{m-1}G(n, 0) + F_m G(n, 1) \\ &= F_{m-1}F_n + F_m(F_n + iF_{n+1}) = (F_{m+1} + F_m)F_n + iF_m F_{n+1}, \end{aligned}$$

and so by (2.1) the complex Fibonacci numbers $G(n, m)$ are given by

$$G(n, m) = F_n F_{m-1} + iF_{n+1}F_m. \tag{2.7}$$

It can be noted at once that along the horizontal axis $G(n, 0) = F_n$, and that on the vertical axis $G(0, m) = iF_m$. Also, the special case $n = 1$ corresponds to the complex numbers considered by Horadam [4].

## 3. RECURRENCE EQUATIONS AND IDENTITIES

Combining (2.2) and (2.3), it follows that

$$G(n + 2, m + 2) = G(n + 1, m + 1) + G(n + 1, m) + G(n, m + 1) + G(n, m), \tag{3.1}$$

which is an interesting two-dimensional version of the Fibonacci recurrence relation and gives the growth-characteristic of the numbers in a diagonal direction: any complex Fibonacci number $G(n, m)$ is the sum of the four previous numbers at the vertices of a square diagonally below and to the left of that number's position on the Gaussian lattice.

From (2.7) and (2.1), it follows that

$$\begin{aligned} G(n + 1, m + 1) &= F_{n+1}F_{m+2} + iF_{n+2}F_{m+1} \\ &= F_{n+1}(F_{m+1} + F_m) + i(F_{n+1} + F_n)F_{m+1} \\ &= F_{n+1}F_{m+1}(1 + i) + F_m F_{n+1} + iF_{m+1}F_n, \end{aligned}$$

and so by (2.7) again, the following recurrence equation is obtained:

$$G(n + 1, m + 1) = (1 + i)F_{n+1}F_{m+1} + G(n, m). \tag{3.2}$$

By repetition of equation (3.2), it follows that

$$G(n + 2, m + 2) = (1 + i)(F_{n+2}F_{m+2} + F_{n+1}F_{m+1}) + G(n, m), \tag{3.3}$$

and by repeated application of (3.2) and (3.3) the following even and odd cases result:

$$G(n + 2k, m + 2k) = (1 + i)\sum_{j=1}^{2k} F_{n+j}F_{m+j} + G(n, m), \tag{3.4}$$

$$G(n + 2k + 1, m + 2k + 1) = (1 + i)\sum_{j=1}^{2k+1} F_{n+j}F_{m+j} + G(m, n). \tag{3.5}$$

From (3.4),

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = (1 + i)^{-1}[G(n + 2k, m + 2k) - G(n, m)],$$

and so by (2.7),

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = \frac{1}{2}(1 - i)[F_{n+2k}F_{m+2k+1} - F_nF_{m+1} + iF_{n+2k+1}F_{m+2k} - iF_{n+1}F_m],$$

and, equating real and imaginary parts

$$F_{n+2k}F_{m+2k+1} - F_{n+2k+1}F_{m+2k} + F_{n+1}F_m - F_nF_{m+1} = 0, \tag{3.6}$$

and

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = \frac{1}{2}[F_{n+2k}F_{m+2k+1} - F_nF_{m+1} + F_{n+2k+1}F_{m+2k} - F_{n+1}F_m]. \tag{3.7}$$

Substitution for $F_{n+2k}F_{m+2k+1}$ from (3.6) into (3.7) gives

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = F_{n+2k+1}F_{m+2k} - F_{n+1}F_m. \tag{3.8}$$

Similarly, for the odd case,

$$\sum_{j=1}^{2k+1} F_{n+j}F_{m+j} = F_{n+2k+2}F_{m+2k+1} - F_nF_{m+1}. \tag{3.9}$$

Identities (3.8) and (3.9) unify and generalize certain identities of Berzsenyi [1] and provide interesting examples as special cases. For example, $n = m = 0$ yields the well-known identity:

$$F_1^2 + F_2^2 + \cdots + F_N^2 = F_NF_{N+1}.$$

From (3.8), the case $m = 0$, $n = 1$ gives

$$F_1F_2 + F_2F_3 + \cdots + F_{2k}F_{2k+1} = F_{2k}F_{2k+2},$$

and from (3.9), $n = 0$, $m = 1$ gives the identity

$$F_1F_2 + F_2F_3 + \cdots + F_{2k+1}F_{2k+2} = F_{2k+2}^2.$$

Many other interesting identities can be specified in this way by suitable choice of parameters. For example, equation (3.8) with $m = 0$, $n = 2$ gives

$$F_1F_3 + F_2F_4 + \cdots + F_{2k}F_{2k+2} = F_{2k}F_{2k+3},$$

and for $m = 2$, $n = 0$, equation (3.9) gives

$$F_1F_3 + F_2F_4 + \cdots + F_{2k+1}F_{2k+3} = F_{2k+2}F_{2k+3}.$$

Identity (3.6) has the following counterpart for the case $2k + 1$:

$$F_{n+2k+1}F_{m+2k+2} - F_{n+2k+2}F_{m+2k+1} = F_mF_{n+1} - F_{m+1}F_n, \tag{3.10}$$

and together (3.6) and (3.10) constitute a generalization of some well-known classical identities. For example, if $n = 1$, $m = 0$, they give

$$F_{N-1}F_{N+1} - F_N^2 = (-1)^N, \; N \geq 1.$$

As another example, equations (3.6) and (3.10) with $n = 1$ and $m = -2$ yield the identity

$$F_{N-1}F_{N+1} - F_{N-2}F_{N+2} = 2(-1)^N.$$

## 4. HIGHER DIMENSIONS

The above development of complex Fibonacci numbers naturally extends to higher dimensions and, in order to illustrate, the three-dimensional case will be outlined.

The number $G(\ell, m, n)$ will be required to satisfy

$$G(\ell + 2, m, n) = G(\ell + 1, m, n) + G(\ell, m, n), \tag{4.1}$$

$$G(\ell, m + 2, n) = G(\ell, m + 1, n) + G(\ell, m, n), \tag{4.2}$$

and

$$G(\ell, m, n + 2) = G(\ell, m, n + 1) + G(\ell, m, n), \tag{4.3}$$

where

$$G(0, 0, 0) = (0, 0, 0);\ G(1, 0, 0) = (1, 0, 0);\ G(0, 1, 0) = (0, 1, 0);$$

$$G(0, 0, 1) = (0, 0, 1);\ G(1, 1, 0) = (1, 1, 0);\ G(1, 0, 1) = (1, 0, 1);$$

$$G(1, 1, 1) = (1, 1, 1).$$

Thus, $G$ has a Fibonacci recurrence in each of the three coordinate directions.

Each of (4.1), (4.2), and (4.3) is a generalized Fibonacci sequence; thus, from (4.1),

$$G(\ell, 0, 0) = F_{\ell - 1}(0, 0, 0) + F_\ell(1, 0, 0) \tag{4.4}$$

and from (4.1) again,

$$G(\ell, 1, 0) = F_{\ell - 1}(0, 1, 0) + F_\ell(1, 1, 0). \tag{4.5}$$

From (4.2), it follows that

$$G(\ell, m, 0) = F_{m-1}G(\ell, 0, 0) + F_m G(\ell, 1, 0). \tag{4.6}$$

From (4.1) again

$$G(\ell, 0, 1) = F_{\ell - 1}(0, 0, 1) + F_\ell(1, 0, 1), \tag{4.7}$$

and

$$G(\ell, 1, 1) = F_{\ell - 1}(0, 1, 1) + F_\ell(1, 1, 1). \tag{4.8}$$

Equation (4.2) then gives

$$G(\ell, m, 1) = F_{m-1}G(\ell, 0, 1) + F_m G(\ell, 1, 1), \tag{4.9}$$

and from (4.3),

$$G(\ell, m, n) = F_{n-1}G(\ell, m, 0) + F_n G(\ell, m, 1). \tag{4.10}$$

Combining equations (4.4)-(4.10), and using the classical Fibonacci recurrence to reduce the expressions obtained, one finally gets

$$G(\ell, m, n) = (F_\ell F_{m+1} F_{n+1},\ F_{\ell+1} F_m F_{n+1},\ F_{\ell+1} F_{m+1} F_n),$$

which is the three-dimensional version of Fibonacci numbers. This form readily generalizes to higher dimensions in the obvious fashion.

It is interesting to note that if (4.1), (4.2), and (4.3) are combined directly, then it follows that the value of $G(\ell + 2, m + 2, n + 2)$ is given by the sum of the values of $G$ at the eight vertices of the cube diagonally below that point—a generalization of (3.1).

The structure provided by the complex Fibonacci numbers was seen in Section 3 to result in some interesting classical identities involving products. It is conjectured that the above three-dimensional numbers may lead to identities involving triple products.

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