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ON THE EQUATION $\sigma(m)\sigma(n) = (m + n)^2$

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1. A pair of positive integers m and n are called amicable if

$$\sigma(m)\sigma(n) = (m + n)^2 \quad \text{and} \quad \sigma(m) = \sigma(n).$$

Although over a thousand pairs of amicable numbers are known, no pairs of relatively prime amicable numbers are known. Some necessary conditions for existence of such numbers are given in [1], [2], and [3].

In this paper, we show that some of the conditions are also necessary for the existence of m and n satisfying

$$\sigma(m)\sigma(n) = (m + n)^2, \tag{1}$$

and

$$(m, n) = 1. \tag{2}$$

In particular we prove

Theorem: If m and n satisfy (1) and (2), mn is divisible by at least twenty-two distinct primes.

Corollary (Hagis [3]): The product of relatively prime amicable numbers are divisible by twenty-two distinct primes.

2. Throughout this paper, let m and n be positive integers satisfying (1) and (2), and let

$$mn = \prod_{i=1}^{r} p_i^{a_i}$$

where $p_1 < \cdots < p_r$ are primes and the a_i's are positive integers. Since σ is multiplicative,

$$\prod_{i=1}^{r} \sigma(p_i^{a_i}) = \sigma(mn) = (m + n)^2.$$

If k and a are positive integers, p is a prime and if $p^a | k$ and $p^{a+1} \nmid k$, then we write $p^a || k$. $\omega(k)$ denotes the number of distinct prime factors of k.

Lemma 1: $\sigma(mn)/mn > 4$.

Proof: By (1) and (2)

$$\frac{\sigma(mn)}{mn} = \frac{(m + n)^2}{mn} = 4 + \frac{(m - n)^2}{mn} > 4. \quad \text{Q.E.D.}$$

Lemma 2: If q is a prime, $q | mn$ and if $p^a || mn$, $q \nmid \sigma(p^a)$.

Proof: Suppose q is a prime, $q | mn$, $p^a || mn$, and $q | \sigma(p^a)$. Since

$$\sigma(p^a) \,|\, (m + n)^2,$$

$q | m + n$. Then $q | m$ and $q | n$, contradicting (2). Q.E.D.

Lemma 3: If $\omega(mn) \le 21$, $2|mn$.

Proof: Suppose

$$mn = \prod_{i=1}^{r} p_i^{a_i}, \quad r \le 21, \text{ and } 3 \le p_1.$$

If q_i is the ith prime, we have, by Lemma 1,

$$4 < \frac{\sigma(mn)}{mn} = \prod_{i=1}^{r} \frac{\sigma(p_i^{a_i})}{p_i^{a_i}} < \prod_{i=1}^{r} \frac{p_i}{p_i - 1} \le \prod_{i=2}^{r+1} \frac{q_i}{q_i - 1}.$$

Since

$$\prod_{i=2}^{r+1} q_i/(q_i - 1) < 4 \text{ if } r \le 20, \; r = 21.$$

Then

(3) $$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 | mn$$

and

(4) $$p_{21} \le 113$$

because

$$\prod_{i=2}^{17} \frac{q_i}{q_i - 1} \prod_{i=19}^{22} \frac{q_i}{q_i - 1} < 4 \quad \text{and} \quad \prod_{i=2}^{21} \frac{q_i}{q_i - 1} \frac{127}{126} < 4.$$

Suppose $p^d || mn$ and $p \ne 3, 7, 31$. Then $p \le 113$ and $p \equiv -1(q)$ for some prime $3 \le q \le 37$. If d is odd, then $1 + p | \sigma(p^d)$, and we have $q | \sigma(p^d)$ and $q | mn$, contradicting Lemma 2. Hence, d is even, and $mn = 3^a 7^b 31^c e^2$, where

$$(e, 2 \cdot 3 \cdot 7 \cdot 31) = 1.$$

Since

$$\prod_{i=1}^{21} \sigma(p_i^{a_i}) = (m+n)^2$$

is even and $\sigma(p_i^{a_i})$ is odd if a_i is even, at least one of a, b, or c is odd.

Suppose at least two of them are odd. Then

$$32 | \sigma(3^a)\sigma(7^b)\sigma(31^c)\sigma(e^2) = (m+n)^2,$$

or $8 | m + n$. Hence, $m \equiv -n(8)$, or $mn = -n^2 \equiv -1(8)$. If a is even, then b and c are odd and $mn = 3^a 7^b 31^c e^2 \equiv 1(8)$, while, if a is odd, then $mn \equiv \pm 3(8)$, a contradiction in both cases. Hence, only one of a, b, or c is odd.

Suppose a is odd and b and c are even. Then

$$m = 3^a f^2 \equiv 3(8)$$

and

$$n = g^2 \equiv 1(8) \quad [\text{or } m \equiv 1(8) \text{ and } n = 3(8)].$$

Hence, $m = 3 + 8h$ and $n = 1 + 8i$, for some h and i, and we have

$$\sigma(3^a)\sigma(f^2 g^2) = (m+n)^2 = (3 + 8h + 1 + 8i)^2 \equiv 0(16),$$

or $16 | \sigma(3^a)$. Since

$$\sigma(3^a) = (1 + 3)(1 + 3^2 + 3^4 + \cdots + 3^{a-1}),$$

$4 | 1 + 3^2 + 3^4 + \cdots + 3^{a-1}$, or $a \equiv 7(8)$. Then

$$\sigma(3^a) = (1 + 3)(1 + 3^2)(1 + 3^4)(1 + 3^8 + \cdots + 3^{a-7}),$$

or $5 | \sigma(3^a)$ contradicting Lemma 2. Suppose b is odd and a and c are even. Then $m \equiv 7(8)$ and $n \equiv 1(8)$ [or $m \equiv 1(8)$ and $n = 7(8)$], $(m+n)^2 \equiv 0(64)$, $64 | \sigma(7^b)$, $b \equiv 7(8)$, $1 + 7^2 | \sigma(7^b)$, or $5 | \sigma(7^b)$, a contradiction. Suppose c is odd and a and

b are even. Then $64|\sigma(31^e)$, $e \equiv 3(4)$, $1 + 31^2|\sigma(31^e)$, or $13|\sigma(31^e)$, a contradiction. Since we get a contradiction in every case,

$$2|mn \text{ if } \omega(mn) \leq 21. \quad \text{Q.E.D.}$$

Lemma 4: $4\nmid mn$.

Proof: Suppose

$$mn = 2^a \prod_{i=2}^{r} p_i^{a_i}, \text{ with } a \geq 1.$$

Since

$$\sigma(mn) = (2^{a+1} - 1)\prod_{i=2}^{r} \sigma(p_i^{a_i}) = (m+n)^2$$

is odd, a_i is even, and we have

$$m = 2^a b^2 \text{ and } n = c^2 \quad [\text{or } m = c^2 \text{ and } n = 2^a b^2].$$

Suppose a is even. Then $m = d^2$, and $2^{a+1} - 1$ has a prime factor $q \equiv 3(4)$. Since $q|m+n$, $c^2 \equiv -d^2(q)$. Since $(q, dc) = 1$ by Lemma 2, $(-d^2/q) = 1$, where (e/q) is the Legendre symbol. However,

$$(-d^2/q) = (-1/q) = (-1)^{\frac{q-1}{2}} = -1,$$

a contradiction. Hence, a is odd.

Suppose $a \geq 3$ is odd. Then

$$m = 2b^2 \text{ and } n = c^2 \quad [\text{or } m = c^2 \text{ and } n = 2b^2],$$

and $2^{a+1} - 1$ has a prime factor $q \equiv 5$ or $7(8)$. Since $q|m+n$ and $(q, 2bc) = 1$, $c^2 \equiv -2b^2(q)$, or $(-2b^2/q) = 1$. However,

$$(-2b^2/q) = (-2/q) = (-1/q)(2/q) = (-1)^{\frac{q-1}{2}}(-1)^{\frac{q^2-1}{8}} = -1,$$

a contradiction, Hence, $a = 1$. Q.E.D.

Lemma 5: If $2|mn$, $\omega(mn) \geq 22$.

Proof: Suppose

$$mn = 2\prod_{i=2}^{r} p_i^{a_i} \quad \text{and} \quad r \leq 21.$$

Since

$$3\prod_{i=2}^{r} \sigma(p_i^{a_i}) = (m+n)^2$$

is odd, $3\nmid mn$, by Lemma 2, and so $5 \leq p_i$, a_i is even, and $3|\sigma(p_j^{a_j})$ for some j. Then, as in Lemma 3, we have $r = 21$, (3) and (4). We can also show that $p_{20} \leq 83$. Suppose $p^a||mn$, $p \geq 5$, q is a prime, and $q|\sigma(p^a)$. Then, by Lemma 2, $q\nmid mn$, and by (3), $q > 61$; moreover, since $q|m+n$,

$$m = 2b^2 \text{ and } n = c^2 \quad [\text{or } m = c^2 \text{ and } n = 2b^2],$$

we have $c^2 \equiv -2b^2(q)$, or $(-2b^2/q) = 1$, and so $q \equiv 1$ or $3(8)$. Hence, if

(5) $$5 \leq q \leq 61, \text{ or } q \equiv 5 \text{ or } 7(8),$$

$q\nmid\sigma(p^a)$.

In [3] Hagis showed that if $3|\sigma(p^a)$ then $\sigma(p^a)$ is divisible by a prime q satisfying (5), except when $p = 31$, 73, 97, or 103, in which case $\sigma(p^a)$ is divisible by $s = 331$, 1801, 3169, or 3571, respectively. Since

$$3\prod_{i=2}^{r}\sigma(p_i^{a_i}) = (m+n)^2, \quad s^2 \Big| \prod_{i=2}^{r}\sigma(p_i^{a_i}).$$

However, Hagis also showed that, if $t^b || mn$, $t \neq p$ and $s | \sigma(t^b)$, then $\sigma(t^b)$ is divisible by a prime q satisfying (5). Hence, $s^2 | \sigma(p^a)$. Then $\sigma(p^a)$ is divisible by a prime q = 5564773, 13925333, 570421, or 985597, respectively, satisfying (5), a contradiction. Hence, $r > 21$. Q.E.D.

Lemmas 3 and 5 prove our Theorem.

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A SPECIAL *m*TH-ORDER RECURRENCE RELATION

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1. INTRODUCTION

In this paper, we consider mth-order recurrence relations whose characteristic equation has only one distinct root. We express the solution for the relation in powers of the single root. The proof for the solution depends upon a special property of factorial polynomials that is given in the first lemma. We conclude the paper by noting the simple form of the result for $m \geq 2$, 3.

2. A SPECIAL *m*TH-ORDER RECURRENCE RELATION

In this section, we shall consider an mth-order recurrence relation whose characteristic equation has only one distinct root λ. It is of the form

$$T = \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{m-j} \lambda^j T_{n-j}$$

with initial values $T_0, \ldots, T_{m-1}$.

Before we can prove the solution for this relation, we must establish two lemmas. The first lemma gives a useful property of the factorial polynomials. With the second lemma, we obtain an evaluation for more general polynomials. These are actually elements in the vector space $\mathbf{V}_m$ of all polynomials in j of degree less than m. This vector space has a basis that consists of the constant ${}_jP_0 = 1$ and the monic factorial polynomials in j:

$${}_jP_w = \frac{j!}{(j-w)!} = (j-0)(j-1) \ldots (j-(w-1)); \; w = 1, \ldots, m-1.$$

We will make use of the fact that the zeros of these polynomials are the integers 0, ..., $w - 1$. We are now ready to state and prove the first lemma.

Lemma 2.1: For any integers m, w where $0 \leq w < m$,

$$\sum_{j=0}^{m} (-1)^j \binom{m}{m-j} {}_jP_w = 0.$$

We first of all observe that for $w = 0$ the factorial polynomials are just the constant 1. For the summation, we then have: