It is clear from Figure 5 that

$$L_1 + L_2 + \cdots + L_m = L_{m+2} - 3,\ m \geq 1. \tag{23}$$

For the generalized sequence, one would find

$$T_1 + T_2 + \cdots + T_m = T_{m+2} - q,\ m \geq 1. \tag{24}$$

Beginning with a $q \times 1$ (black) rectangle, one can use identity (24) successively for $m = 1, 2, \ldots$ to generate $T_m$-gnomons. A variety of identities for generalized Fibonacci numbers can be observed and discovered by mimicking the procedures followed earlier.

It seems appropriate to conclude with a remark of Brother Alfred Brousseau: "It appears that there is a considerable wealth of enrichment and discovery material in the general area of Fibonacci numbers as related to geometry" [1]. Additional geometry of Fibonacci numbers can be found in Bro. Alfred's article.

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# FIBONACCI AND LUCAS CUBES

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## 1. INTRODUCTION

The Fibonacci numbers are defined by the well-known recursion formulas

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$$

and the Lucas numbers by

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2}.$$

J. H. E. Cohn [2] determined the Fibonacci and Lucas numbers that are perfect squares. R. Finkelstein and H. London [3] gave a rather complicated determination of the cubes in the Fibonacci and Lucas sequences. Diophantine equations whose solutions must be Fibonacci and Lucas cubes occur in C. L. Siegel's proof [7] of H. M. Stark's result that there are exactly nine complex quadratic fields of class number one. This paper presents a simple determination of all Fibonacci numbers $F_n$ of the form $2^a 3^b X^3$ and all Lucas numbers $L_n$ of the form $2^a X^3$.

## 2. PRELIMINARY REDUCTIONS

From the recursion formulas defining the Fibonacci and Lucas numbers, it is easily verified by induction that the sequence of residues of $F_n$ and $L_n$ (mod $p$) are periodic, and in particular that

$$2 \mid F_n \text{ iff } 3 \mid n \tag{1}$$

$$2 \mid L_n \text{ iff } 3 \mid n \tag{2}$$

$$3 \mid F_n \text{ iff } 4 \mid n \tag{3}$$

$$3|L_n \text{ iff } n \equiv 2 \pmod 4 \tag{4}$$

$$5\nmid L_n \tag{5}$$

$$7|L_n \text{ iff } n \equiv 4 \pmod 8 \tag{6}$$

If $\varepsilon_0 = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\varepsilon}_0 = \dfrac{1-\sqrt{5}}{2}$, it is also easily verified by induction that:

$$\varepsilon_0 = \frac{L_n + F_n\sqrt{5}}{2},\ F_n = \frac{1}{\sqrt{5}}(\varepsilon_0^n - \overline{\varepsilon}_0^n),\ L_n = \varepsilon_0^n + \overline{\varepsilon}_0^n.$$

From these formulas, the following identities are easily derived:

$$5F_n^2 - L_n^2 = 4(-1)^{n+1} \tag{7}$$

$$F_{2n} = F_n L_n \tag{8}$$

$$4F_{3n} = F_n(5F_n^2 + 3L_n^2) \tag{9}$$

$$4L_{3n} = L_n(15F_n^2 + L_n^2) \tag{10}$$

Further, from (1), (2), and (7), we find that

$$(F_n, L_n) = \begin{cases} 2 & \text{if } 3|n \\ 1 & \text{otherwise} \end{cases} \tag{11}$$

Finally, since $F_n = (-1)^n F_{-n}$ and $L_n = (-1)^n L_{-n}$, it suffices to consider the case $n > 0$ in what follows.

The identity (7) is the basis of a reduction of the determination of Fibonacci or Lucas cubes (or, more generally, Fibonacci and Lucas $P$th powers) to solving particular Diophantine equations. It turns out that this identity actually characterizes Fibonacci and Lucas numbers, in the sense that $(L_{2n}, F_{2n})$ for $n > 0$ is the complete set of positive solutions to the Diophantine equation $X^2 - 5Y^2 = 4$, and $(L_{2n+1}, F_{2n+1})$ for $n \geq 0$ is the complete set of positive solutions to the Diophantine equation $X^2 - 5Y^2 = -4$. From these facts, it follows that the positive Fibonacci cubes are exactly those positive $Y^3$ for which $X^2 - 5Y^6 = \pm 4$ is solvable in integers, and the positive Lucas cubes are those positive $X^3$ for which $X^6 - 5Y^2 = \pm 4$ is solvable in integers. For our purposes, it suffices to know only that (7) holds, so that the Fibonacci and Lucas cubes are a *subset* of the solutions of these Diophantine equations.

We now show that the addition formulas (8)-(10) can be used to relate Fibonacci numbers of the form $2^a 3^b X^3$ to those of the form $X^3$, and Lucas numbers of the form $2^a X^3$ to those of the form $X^3$.

*Lemma 1*: (i) If $F_{2n}$ is of the form $2^a 3^b X^3$, so is $F_n$.
(ii) If $F_{3n}$ is of the form $2^a 3^b X^3$, so is $F_n$.
(iii) If $L_{3n}$ is of the form $2^a X^3$, so is $L_n$.

*Proof*: (i) follows from (8) and (11). (ii) follows from (9) and (11), where we note that $(F_n, 3L_n^2)|12$. Finally, (iii) follows from (10), (11), and (5), noting that $(L_n, 15F_n^2)|12$.

*Lemma 2*: (i) If $F_n = 2^a 3^b X^3$ and $n = 2^c 3^d k$ with $(6, k) = 1$, then $F_k = Z^3$.
(ii) If $L_n = 2^a X^3$ and $n = 3^d k$ with $(3, k) = 1$, then $L_k = Z^3$.

*Proof*: For (i), note that $F_k$ is of the form $2^a 3^b X^3$ by repeated application of Lemma 1, while $(F_k, 6) = 1$ by (1) and (3), so $F_k = Z^3$. (ii) has a similar proof using (2).

*Remark*: The preceding two lemmas are both valid in the more general case where "cube" is replaced by "$P$th power" throughout, using the same proofs.

## 3. MAIN RESULTS

*Theorem 1*: The only $F_n$ with $(n, 6) = 1$ that are cubes are $F_1 = 1$ and $F_{-1} = -1$.

*Proof*: Let $F_n = Z^3$ and note that $(n, 6) = 1$ and (1) and (7) yield

$$5Z^6 - 4 = L_n^2 \quad \text{and} \quad (2, Z) = 1. \tag{12}$$

Setting $X = 5Z^2$ and $Y = 5L_n$ yields

$$X^3 - 100 = Y^2 \tag{13}$$

and (2) and (4) require $(Y, 6) = 1$. We examine (13) over the ring of integers of $Q(\sqrt[3]{10})$. It has been shown (see [6] and [8]) that this ring has unique factorization, that its members are exactly those $(1/3)(A + B\sqrt[3]{10} + C\sqrt[3]{100})$ where $A$, $B$, and $C$ are integers with $A \equiv B \equiv C \pmod 3$, and that the units in this ring are of the form $\pm\varepsilon^K$ where $\varepsilon = (1/3)(23 + 11\sqrt[3]{10} + 5\sqrt[3]{100})$. Equation (13) factors as

$$(X - \sqrt[3]{100})(X^2 + \sqrt[3]{100}X + 10\sqrt[3]{10}) = Y^2. \tag{14}$$

Write

$$X - \sqrt[3]{100} = \eta\alpha^2, \tag{15}$$

where $\eta$ in square free and divides both $X - \sqrt[3]{100}$ and $X^2 + \sqrt[3]{100} + 10\sqrt[3]{10}$. Then

$$\eta \mid (X^2 + \sqrt[3]{100}X + 10\sqrt[3]{10}) - (X + 2\sqrt[3]{100})(X - \sqrt[3]{100}) = 30\sqrt[3]{10}.$$

Since $(Y, 3) = 1$, $(\eta, 3) = 1$, and $\eta \mid 10\sqrt{10}$. Now $(\sqrt{10})^3 = 2 \cdot 5$ and $(2, 5) = 1$, so by unique factorization we can find $\Delta$ and $\Phi$ such that $\sqrt[3]{10} = \Delta\Phi$, $5 = \Delta^3\varepsilon^K$, and $2 = \Phi^3\varepsilon^{-K}$. Then $\eta \mid 10\sqrt[3]{10} = \Delta^4\Phi^4$. Now $Y = 5L_n$ and $(2, L_n) = 1$ by (2), so (14) shows that $(\Phi, X - \sqrt[3]{100}) = 1$. Hence $\eta \mid \Delta^4$. But $5 \mid X$, so $\Delta^3 \mid X$, and hence $\Delta^2 \| X - \sqrt[3]{100}$. Since $\eta$ is square free, $\eta$ must be a unit. By absorbing squares of units into $\alpha$, we need only consider $\eta = \pm 1$ and $\eta = \pm\varepsilon$ in (15).

*Case 1*: $X - \sqrt[3]{100} = \pm\alpha^2$. Let $\alpha = (1/3)(A + B\sqrt[3]{10} + C\sqrt[3]{100})$. Since representation of integers in this form is unique.

$$X = \pm\frac{1}{9}(A^2 + 20BC) \tag{16}$$

$$0 = \pm\frac{1}{9}(2AB + 10C^2) \tag{17}$$

$$-1 = \pm\frac{1}{9}(B^2 + 2AC) \tag{18}$$

Equation (17) shows $B \mid 5C^2$. Squaring (18) and multiplying both sides by $3^4 \cdot 5$, we see that $B$ divides each term on the right side so $B \mid 3^4 \cdot 5$. For each of the twenty values of $B$ satisfying $B \mid 3^4 \cdot 5$, we can solve (17) and (18) for $A$ and $C$, and verify the only integer solutions $(A, B, C)$ are $(-5, 1, 1)$ and $(5, -1, -1)$ when $\eta = 1$, and $(0, \pm 3, 0)$ when $\eta = -1$. Evaluating $X$ by (16) we find that the first two solutions yield $Z = \pm 1$ in (12), and thus $F_1 = 1$ and $F_{-1} = -1$, while the third solution is extraneous to (13).

*Case 2*: $X - \sqrt[3]{100} = \pm\varepsilon\alpha^2$. Proceeding as in Case 1, we obtain

$$X = \pm\frac{1}{27}(23A^2 + 110B^2 + 500C^2 + 100AB + 220AC + 460BC) \tag{19}$$

$$0 = \pm\frac{1}{27}(11A^2 + 50B^2 + 230C^2 + 46AB + 100AC + 230BC) \tag{20}$$

$$-1 = \pm\frac{1}{27}(5A^2 + 23B^2 + 110C^2 + 22AB + 46AC + 100BC) \tag{21}$$

From (20) $2 \mid A$ so that $2 \mid X$ in (19), and such solutions are extraneous to (12).

*Remark:* It can be shown (see [1] and [3]) that the complete set of solutions $(X, Y)$ to (13) is (5, ±5), (10, ±30), and (34, ±198).

*Theorem 2:* The set of Fibonacci numbers $F_n$ with $n > 0$ of the form $2^a 3^b X^3$ is $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_6 = 8$, and $F_{12} = 144$.

*Proof:* Let $F_n = 2^a 3^b X^3$ with $n = 2^c 3^d k$ and $(k, 6) = 1$. By Lemma 2, $F_k = Z^3$ and by Theorem 1, $k = 1$. If $c \geq 3$, repeated application of Lemma 1(ii) would show $F_8 = 21$ is of the form $2^a 3^b X^3$, which is false. If $d \geq 2$, repeated application of Lemma 1(i) would show $F_9 = 34$ is of the form $2^a 3^b X^3$, which is false. The values $0 \leq c \leq 2$ and $0 \leq d \leq 1$ give the stated solutions.

*Theorem 3:* The equation $L_{2n} = X^3$ has no solutions.

*Proof:* Suppose $L_{2n} = X^3$. Then (7) yields

$$5F_{2n}^2 + 4 = X^6.$$

All solutions to this equation (mod 7) require $7|X$. Then (6) shows $4|2n$ hence $3|F_{2n}$ by (2), so $X^6 \equiv 4 \pmod 9$, which is impossible.

*Theorem 4:* The equation $L_n = X^3$ with $(n, 6) = 1$ has only the solutions $L_1 = 1$ and $L_{-1} = -1$.

*Proof:* Suppose $L_n = X^3$ with $(n, 6) = 1$. Then (2) and (7) yield

$$5F_n^2 - 4 = X^6 \quad \text{and} \quad (6, X) = 1. \tag{22}$$

We examine (22) over the ring of integers of $Q(\sqrt{5})$. It is known that this ring has unique factorization, that these integers are of the general form

$$\frac{1}{2}(A + B\sqrt{5})$$

with $A \equiv B \pmod 2$, and that the units are of the form $\pm\varepsilon_0^K$, where

$$\varepsilon_0 = \frac{1}{2}(1 + \sqrt{5}).$$

Now (22) gives

$$(\sqrt{5}F_n + 2)(\sqrt{5}F_n - 2) = Z^3,$$

where $Z = X^2$. Then

$$\sqrt{5}F_n + 2 = \eta\alpha^3,$$

where $\eta$ divides both $\sqrt{5}F_n + 2$ and $\sqrt{5}F_n - 2$. Then we have $\eta|4$. But $(2, Z) = 1$, so $(2, \sqrt{5}F_n + 2) = 1$ and $\eta$ is a unit. By absorbing cubes of units, we need to consider only $\eta = 1$, $\varepsilon_0$, and $\varepsilon_0^{-1}$.

*Case 1:* $2 + F_n\sqrt{5} = \alpha^3$. Let $\alpha = (1/2)(A + B\sqrt{5})$. Substituting this yields the equations

$$2 = \frac{1}{8}A\,(A^2 + 15B^2)$$
$$F_n = \frac{1}{8}B\,(3A^2 + 5B^2). \tag{23}$$

Then (23) shows that $A|16$ and $|B| \leq 1$, from which $A = 1$ and $B = \pm 1$ are the only solutions, yielding $F_n = \pm 1$ and, finally, $L_1 = 1$ and $L_{-1} = -1$.

*Case 2:* $2 + F_n\sqrt{5} = \varepsilon_0\alpha^3$. Let $\alpha = (1/2)(A + B\sqrt{5})$ with $A \equiv B \pmod 2$, which yields

$$2 = \frac{1}{16}(A^3 + 15A^2B + 15AB^2 + 25B^3)$$

and

$$F_n = \frac{1}{16}(A^3 + 3A^2B + 15AB^2 + 5B^3).$$

Then

$$4(2 - F_n) = B(3A^2 + 5B^2) \equiv 4 \pmod 8,$$

because $2 \nmid F_n$ since $(n, 6) = 1$. This congruence has no solutions with $A \equiv B \pmod 2$.

*Case 3*: $2 + F_n\sqrt{5} = \varepsilon_0^{-1}\alpha^3$. Noting $\varepsilon_0^{-1} = (1/2)(1 - \sqrt{5})$, we argue as in Case 2, using instead

$$4(2 + F_n) = -B(3A^2 + 5B^2) \equiv 4 \pmod 8,$$

which has no solutions with $A \equiv B \pmod 2$.

*Theorem 5*: The set of Lucas numbers $L_n$ with $n > 0$ of the form $2^aX^3$ are $L_1 = 1$ and $L_3 = 4$.

*Proof*: Let $L_n = 2^aX^3$ with $n = 3^ck$ and $(k, 3) = 1$. By Lemma 2, $L_k = X^3$ so by Theorems 3 and 4, $k = 1$. If $c \geq 2$, then Lemma 2(ii) would show $L_9 = 76$ was of the form $2^aX^3$, which is false.

*Remark*: The set of Lucas numbers of the form $2^a3^bX^3$ leads to consideration of the equation $X^3 = Y^2 + 18$. The only solutions to this equation are $(3, \pm 3)$, but the available proofs (see [1] and [3]) are complicated. General methods for solving the equation $X^3 = Y^2 + K$ for fixed $K$ are given in [1], [4], and [5].

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# THE NUMBER OF STATES IN A CLASS OF SERIAL QUEUEING SYSTEMS


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## ABSTRACT


It is shown that the number of states in a class of serial production or service systems with $N$ servers is the $(2N - 1)$st Fibonacci number. This has proved useful in designing efficient systems.


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