THE DECIMAL EXPANSION OF 1/89 AND RELATED RESULTS

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One of the more bizarre and unexpected results concerning the Fibonacci sequence is the fact that

$$\begin{aligned}\frac{1}{89} = .0112358&\\ 13&\\ 21&\\ 34&\\ 55&\\ 89&\\ 144&\\ 233&\\ \ldots&\\ = \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^i},&\end{aligned} \tag{1}$$

where F_i denotes the ith Fibonacci number. The result follows immediately from Binet's formula, as do the equations

$$\frac{19}{89} = \sum_{i=1}^{\infty} \frac{L_{i-1}}{10^i} \tag{2}$$

$$\frac{1}{109} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-10)^i} \tag{3}$$

and

$$-\frac{21}{109} = \sum_{i=1}^{\infty} \frac{L_{i-1}}{(-10)^i}. \tag{4}$$

where L_i denotes the ith Lucas numbers. It is interesting that all these results can be obtained from the following unusual identity, which is easily proved by mathematical induction.

Theorem 1: Let a, b, c, d, and B be integers. Let $\{\mu_n\}$ be the sequence defined by the recurrence $\mu_0 = c$, $\mu_1 = d$, $\mu_{n+2} = a\mu_{n+1} + b\mu_n$ for all $n \geq 2$. Let m and N be integers defined by the equations

$$B^2 = m + Ba + b \quad \text{and} \quad N = cm + dB + bc.$$

Then

$$B^n N = m\sum_{i=1}^{n+1} B^{n+1-i}\mu_{i-1} + B\mu_{n+1} + b\mu_n \tag{5}$$

for all $n \geq 0$. Also, $N \equiv 0 \pmod{B}$.

Proof: The result is clearly true for $n = 0$, since it then reduces to the equation

$$N = cm + dB + bc$$

of the hypotheses. Assume that

$$B^k N = m\sum_{i=1}^{k+1} B^{k+1-i}\mu_{i-1} + B\mu_{k+1} + b\mu_k.$$

Then

$$B^{k+1} N = m\sum_{i=1}^{k+1} B^{k+2-i}\mu_{i-1} + B^2\mu_{k+1} + Bb\mu_k$$

$$= m\sum_{i=1}^{k+1} B^{k+2-i}\mu_{i-1} + (m + Ba + b)\mu_{k+1} + Bb\mu_k$$

$$= m\sum_{i=1}^{k+2} B^{k+2-i}\mu_{i-1} + B(a\mu_{k+1} + b\mu_k) + b\mu_{k+1}$$

$$= m\sum_{i=1}^{k+2} B^{k+2-i}\mu_{i-1} + B\mu_{k+2} + b\mu_{k+1}.$$

This completes the induction. Finally, to see that $N \equiv 0 \pmod{B}$, we have only to note that

$$N = cm + dB + bc = c(B^2 - Ba - b) + dB + bc = cB^2 - caB + dB \equiv 0 \pmod{B}.$$

Now, it is well known that the terms of the sequence defined in Theorem 1 are given by

$$\mu_n = \left(\frac{c}{2} + \frac{2d - c}{\sqrt{a^2 + 4b}}\right)\left(\frac{a + \sqrt{a^2 + 4b}}{2}\right)^n + \left(\frac{c}{2} - \frac{2d - c}{\sqrt{a^2 + 4b}}\right)\left(\frac{a - \sqrt{a^2 + 4b}}{2}\right)^n. \quad (6)$$

Thus it follows from (5) that

$$\frac{N}{Bm} = \sum_{i=1}^{n+1} \frac{\mu_{i-1}}{B^i} + \frac{B\mu_{n+1} + b\mu_n}{mB^{n+1}} = \sum_{i=1}^{\infty} \frac{\mu_{i-1}}{B^i}, \quad (7)$$

provided that the remainder term tends to 0 as n tends to infinity, and a sufficient condition for this is that

$$\left|\frac{a + \sqrt{a^2 + 4b}}{2B}\right| < 1 \quad \text{and} \quad \left|\frac{a - \sqrt{a^2 + 4b}}{2B}\right| < 1.$$

Thus we have proved the following theorem.

Theorem 2: If a, b, c, d, m, N, and B are integers, with m and N as defined above and if

$$\left|\frac{a + \sqrt{a^2 + 4b}}{2B}\right| < 1 \quad \text{and} \quad \left|\frac{a - \sqrt{a^2 + 4b}}{2B}\right| < 1,$$

then

$$\frac{N}{Bm} = \sum_{i=1}^{\infty} \frac{\mu_{i-1}}{B^i}. \quad (8)$$

Of course, equations (1)-(4) all follow from (8) by particular choices of a, b, c, and d. To obtain (2), for example, we set $c = 2$, $a = b = d = 1$, and $B = 10$. It then follows that

$$m = B^2 - Ba - b = 100 - 10 - 1 = 89$$

$$N = cm + dB + bc = 178 + 10 + 2 = 190$$

and

$$\frac{19}{89} = \frac{190}{10 \cdot 89} = \frac{N}{Bm} = \sum_{i=1}^{\infty} \frac{L_{i-1}}{10^i} \quad \text{as claimed.}$$

To obtain (3), we set $c = 0$, $a = b = d = 1$, and $B = -10$. Then

$$m = B^2 - Ba - b = 100 + 10 - 1 = 109,$$

$$N = cm + dB + bc = -10,$$

and

$$\frac{N}{Bm} = \frac{-10}{-10 \cdot 109} = \frac{1}{109} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-10)^i} \quad \text{as indicated.}$$

Finally, we note that interesting results can be obtained by setting B equal to a power of 10. For example, if $B = 10^h$ for some integer h, $c = 0$, and $a = b = d = 1$,

$$m = 10^{2h} - 10^h - 1,\ N = 10^h,$$

and (8) reduces to

$$\frac{1}{10^{2h} - 10^h - 1} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^{hi}}. \tag{9}$$

For successive values of h this gives

$$\frac{1}{89} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^i} \tag{10}$$

as we already know,

$$\frac{1}{9899} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^{2i}} = .000101020305081321\ldots, \tag{11}$$

$$\frac{1}{998999} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{10^{3i}} = .000001001002003005008013\ldots, \tag{12}$$

and so on. In case $B = (-10)^h$ for successive values of h, $c = 0$, and $a = b = d = 1$, we obtain

$$\frac{1}{109} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-10)^i}, \tag{13}$$

$$\frac{1}{10099} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-100)^i}, \tag{14}$$

$$\frac{1}{1000999} = \sum_{i=1}^{\infty} \frac{F_{i-1}}{(-1000)^i}, \tag{15}$$

and so on. Other fractions corresponding to (2) and (3) above are

$$\frac{19}{89}, \frac{199}{9899}, \frac{1999}{998999}, \ldots$$

and

$$-\frac{21}{109}, -\frac{201}{10099}, -\frac{2001}{1000999}, \ldots .$$

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