This result has been most useful in developing numerical procedures for calculating or approximating the probabilities that a server is busy, which is used in finding efficient designs for this class of production systems.

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THE DETERMINATION OF ALL DECADIC KAPREKAR CONSTANTS

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0. INTRODUCTION

Choose a to be any r-digit integer expressed in base 10 with not all digits equal. Let a' be the integer formed by arranging these digits in descending order, and let a'' be the integer formed by arranging these digits in ascending order. Define $T(a) = a' - a''$. When $r = 3$, repeated applications of T to any starting value a will always lead to 495, which is self-producing under T, that is, $T(495) = 495$. Any r-digit integer exhibiting the properties that 495 exhibits in the 3-digit case will be called a "Kaprekar constant." It is well known (see [2]) that 6174 is such a Kaprekar constant in the 4-digit case.

In this paper we concern ourselves only with self-producing integers. After developing some general results which hold for any base g, we then characterize all decadic self-producing integers. From this it follows that the only r-digit Kaprekar constants are those given above for $r = 3$ and 4.

1. THE DIGITS OF $T(a)$

Let $r = 2n + \delta$, where

$$\delta = \begin{cases} 1 & r \text{ odd} \\ 0 & r \text{ even.} \end{cases}$$

Let a be an r-digit g-adic integer of the form

$$a = \alpha_{r-1}g^{r-1} + \alpha_{r-2}g^{r-2} + \cdots + \alpha_1 g + \alpha_0 \tag{1.1}$$

with

$$g > \alpha_{r-1} \geq \alpha_{r-2} \geq \cdots \geq \alpha_1 \geq \alpha_0,\ \alpha_{r-1} > \alpha_0.$$

Let a' be the corresponding reflected integer

$$a' = \alpha_0 g^{r-1} + \alpha_1 g^{r-2} + \cdots + \alpha_{r-2}g + \alpha_{r-1}. \tag{1.2}$$

The operation $T(a) = a - a'$ will give rise to a new r-digit integer (permitting leading zeros) whose digits can be arranged in descending and ascending order as in (1.1) and (1.2). Define

$$d_{n-i+1} = \alpha_{r-i} - \alpha_{i-1},\ i = 1, 2, \ldots, n. \tag{1.3}$$

Thus associated with the integer a given in (1.1) is the n-tuple of differences $D = (d_n, d_{n-1}, \ldots, d_1)$ with $g > d_n \geq d_{n-1} \geq \cdots \geq d_1$. Note that $T(a)$ depends entirely upon the values of these differences. The digits of $T(a)$ are given by the following, viz.,

$$\delta = 0 \text{ and } d_1 \neq 0 \tag{1.4a}$$

$$d_n \quad d_{n-1} \ldots d_2 \quad d_1 - 1 \quad g - d_1 - 1 \quad g - d_2 - 1 \ldots g - d_{n-1} - 1 \quad g - d_n$$

$$\delta = 0 \text{ and } d_1 = d_2 = \cdots = d_{j-1} = 0,\ d_j \neq 0,\ 1 < j \leq n \tag{1.4b}$$

$$d_n \quad d_{n-1} \ldots d_{j+1} \quad d_j - 1 \quad \overbrace{g - 1 \ldots g - 1}^{2(j-1) \text{ terms}} \quad g - d_j - 1 \ldots g - d_{n-1} - 1 \quad g - d_n$$

$$\delta = 1 \text{ and } d_1 \neq 0 \tag{1.4c}$$

$$d_n \quad d_{n-1} \ldots d_2 \quad d_1 - 1 \quad g - 1 \quad g - d_1 - 1 \quad g - d_2 - 1 \ldots g - d_{n-1} - 1 \quad g - d_n$$

$$\delta = 1 \text{ and } d_1 = d_2 = \cdots = d_{j-1} = 0,\ d_j \neq 0,\ 1 < j \leq n \tag{1.4d}$$

$$d_n \quad d_{n-1} \ldots d_{j+1} \quad d_j - 1 \quad \overbrace{g - 1 \ldots g - 1}^{2(j-1)+1 \text{ terms}} \quad g - d_j - 1 \ldots g - d_{n-1} - 1 \quad g - d_n$$

Differences $D' = (d'_n, d'_{n-1}, \ldots, d'_1)$ can now be assigned to the integers $T(a)$ as in (1.3). We say that $(d_n, d_{n-1}, \ldots, d_1)$ is mapped to $(d'_n, d'_{n-1}, \ldots, d'_1)$ under T.

2. PROPERTIES OF ONE-CYCLES

We shall focus attention on the determination of all a such that $T(a) = a$. Such integers are said to generate a one-cycle a. This is equivalent to finding all n-tuples $(d_n, d_{n-1}, \ldots, d_1)$ that are mapped to themselves under T.

Theorem 2.1: Suppose $(d_n, d_{n-1}, \ldots, d_1)$ represents a one-cycle with $d_j \neq 0$, $j \geq 1$, and $d_k = 0$ for $k < j$. Further suppose that $d_n \neq d_j$. Then

(i) $d_n + d_j = g$ if $\delta = 1$ or if $\delta = 0$ and $j > 1$,

or

(ii) $\begin{cases} d_n + 2d_1 = g \\ \quad\text{or} \\ d_n = g - 1,\ d_1 = 1 \end{cases}$ if $\delta = 0$ and $j = 1$

Proof: (i) Since either $j > 1$ or $\delta = 1$, (1.4a) does not apply. Thus the largest digit in $T(a)$ is $g - 1$. The smallest digit could be one of three:

$$\begin{cases} d_j - 1 & \text{if } d_j + d_n - 1 < g \\ g - d_n & \text{if } d_j + d_n - 1 \geq g,\ d_n \neq d_{n-1} \\ g - d_n - 1 & \text{if } d_j + d_n - 1 \geq g,\ d_n = d_{n-1}. \end{cases}$$

Therefore,

$$d'_n = \begin{cases} g - d_j & \text{if } d_j + d_n - 1 < g \\ d_n - 1 & \text{if } d_j + d_n - 1 \geq g,\ d_n \neq d_{n-1} \\ d_n & \text{if } d_j + d_n - 1 \geq g,\ d_n = d_{n-1}. \end{cases}$$

Since $d_n = d'_n$, if $d_j + d_n - 1 < g$, then $d_n + d_j = g$. If $d_j + d_n - 1 \geq g$, then since $d'_n = d_n \neq d_n - 1$, it must be that $d_n = d_{n-1}$. This condition restricts the second largest digit to be either d_n or $g - 1$, and the second smallest to be $g - d_n$ if $d_n \neq d_{n-2}$ or $g - d_n - 1$ if $d_n = d_{n-2}$. Since $d'_{n-1} = d_{n-1} = d_n \neq g$, we must have $d_n = d_{n-2}$. Continuing in this fashion, one finds that $d_n = d_j$, which contradicts the hypothesis. Thus $d_n + d_j = g$.

(ii) Suppose first that $d_n > g - d_1 - 1$, then d_n is the largest digit in (1.4a). Then

$$d'_n = \begin{cases} d_n - d_1 + 1 & \text{if } d_1 + d_n - 1 < g \\ 2d_n - g & \text{if } d_1 + d_n - 1 \geq g,\ d_n \neq d_{n-1} \\ 2d_n - g + 1 & \text{if } d_1 + d_n - 1 \geq g,\ d_n = d_{n-1}. \end{cases}$$

If $d_1 + d_n - 1 < g$ and $g < d_1 + d_n + 1$, then $g = d_1 + d_n$. Since $d'_n = d_n$, one must have $d_1 = 1$ and $d_n = g - 1$. If $d_1 + d_n - 1 \geq g$, then $d_n = d_{n-1}$ as shown in (i). Hence $d_n = d_{n-1} = \cdots = d_1 = g - 1$. This cannot occur in a one-cycle unless $g = 2$, in which case $d_n = g - 1 = 1 = d_1$. Thus, if $d_n > g - d_1 - 1$, $d_n = g - 1$ and $d_1 = 1$.

Now suppose that $d_n \leq g - d_1 - 1$. Then the largest digit in (1.4a) is $g - d_1 - 1$ and the smallest is $d_1 - 1$. Hence

$$d_n = d'_n = (g - d_1 - 1) - (d_1 - 1) = g - 2d_1$$

and

$$d_n + 2d_1 = g.$$

Theorem 2.2: If $D = (d_n, d_{n-1}, \ldots, d_1)$ represents a one-cycle with $d_n = \cdots = d_j \neq 0$, $j \geq 1$, and $d_k = 0$ for $k < j$, then $d_n = \cdots = d_j = g/2$. Further,

(i) if $g \neq 2$, then $r \equiv 0 \pmod 3$ and $g \equiv 0 \pmod 2$. In particular

$$D = \begin{cases} \Bigl(\overbrace{\tfrac{g}{2}, \tfrac{g}{2}, \ldots, \tfrac{g}{2}}^{r/3 \text{ terms}}, \overbrace{0, 0, \ldots, 0}^{r/6 \text{ terms}}\Bigr) & \text{when } r \equiv 0 \pmod 2 \\ \Bigl(\overbrace{\tfrac{g}{2}, \tfrac{g}{2}, \ldots, \tfrac{g}{2}}^{r/3 \text{ terms}}, \overbrace{0, 0, \ldots, 0}^{(r-3)/6 \text{ terms}}\Bigr) & \text{when } r \equiv 1 \pmod 2 \end{cases}$$

(ii) if $g = 2$, then every n-tuple D is a one-cycle.

Proof: (i) If $g > 2$, then $j > 1$ from (1.4). From (1.4b) and (1.4d), any n-tuple $(k, k, \ldots, k, 0, 0, \ldots, 0)$ will give rise to a successor with digits

$$\overbrace{k\ k \ldots k}^{(n-j) \text{ terms}}\ k - 1\ \overbrace{g - 1 \ldots g - 1}^{2(j-1)+\delta \text{ terms}}\ \overbrace{g - k - 1 \ldots g - k - 1}^{(n-j) \text{ terms}}\ g - k.$$

Clearly the largest digit is $g - 1$. The smallest is either $k - 1$, forcing $k = g/2$, or $g - k - 1$, forcing $k - (g - k) = k$, which is impossible. Hence

$$d_n = d_{n-1} = \cdots = d_j = \frac{g}{2}.$$

Consider

$$D = \Bigl(\overbrace{\tfrac{g}{2}, \tfrac{g}{2}, \ldots, \tfrac{g}{2}}^{a \text{ terms}}, \overbrace{0, 0, \ldots, 0}^{(n-a) \text{ terms}}\Bigr),\ a = n - j + 1.$$

The digits of the successor of D are

$$\overbrace{\tfrac{g}{2}\ \tfrac{g}{2} \cdots \tfrac{g}{2}}^{(a-1) \text{ terms}}\ \tfrac{g}{2} - 1\ \overbrace{g - 1 \ldots g - 1}^{2(n-a)+\delta \text{ terms}}\ \overbrace{\tfrac{g}{2} - 1 \ldots \tfrac{g}{2} - 1}^{(a-1) \text{ terms}}\ \tfrac{g}{2}. \tag{2.1}$$

Ordering the digits of (2.1) in descending order, one obtains

$$\overbrace{g - 1 \ldots g - 1}^{2(n-a)+\delta \text{ terms}}\ \overbrace{\tfrac{g}{2} \cdots \tfrac{g}{2}}^{a \text{ terms}}\ \overbrace{\tfrac{g}{2} - 1 \ldots \tfrac{g}{2} - 1}^{a \text{ terms}}. \tag{2.2}$$

Differences equal to $g/2$ will be generated by the pairs $(g - 1, g/2 - 1)$, and differences will be generated by the pairs $(g/2, g/2)$. Hence, if D is a one-cycle, then $2(n - a) + \delta = a$, that is, $r = 2n + \delta = 3a$. In addition,

$$n - a = \begin{cases} \frac{r}{6} & \text{if } r \equiv 0 \pmod 2 \\ \frac{r-3}{6} & \text{if } r \equiv 1 \pmod 2). \end{cases}$$

(ii) If $g = 2$, then the digits of the successor of D ordered in descending order, from (2.2), are

$$\overbrace{1\ 1\ \ldots\ 1}^{2(n-a)+\delta \text{ terms}} \quad \overbrace{1\ \ldots\ 1}^{a \text{ terms}} \quad \overbrace{0\ \ldots\ 0}^{a \text{ terms}}. \tag{2.3}$$

Clearly the first a succeeding differences in (2.3) are equal to 1 and the remaining $(n-a)$ differences are equal to 0. Therefore, a is a one-cycle for all $1 \le a \le n$.

Definition 2.1: For $i = 0, 1, \ldots, g - 1$, let ℓ_i be the number of entries in $(d_n, d_{n-1}, \ldots, d_1)$ that equal i, and let c_i be the number of digits of $T(a)$ that equal i.

For example, if $g = 10$, $\delta = 0$, and $D = (9, 9, 7, 7, 3, 1, 0, 0)$, then

$\ell_9 = 2$, $\ell_8 = 0$, $\ell_7 = 2$, $\ell_6 = \ell_5 = \ell_4 = 0$, $\ell_3 = 1$, $\ell_2 = 0$, $\ell_1 = 1$, and $\ell_0 = 2$

From (1.4), the digits of D' are

9 9 7 7 3 0 9 9 9 9 8 6 2 2 0 1

giving rise to the digit counters

$c_9 = 6$, $c_8 = 1$, $c_7 = 2$, $c_6 = 1$, $c_5 = c_4 = 0$, $c_3 = 1$, $c_2 = 2$, $c_1 = 1$, and $c_0 = 2$

Using the results of Section 1, we now obtain the following corollary.

Corollary 2.1: If $d_n + d_j = g$, where d_j is the smallest nonzero entry in

$$D = (d_n, d_{n-1}, \ldots, d_1),$$

then

$$c_{g-1} = \ell_{g-1} + 2\ell_0 + \delta$$
$$c_i = \ell_i + \ell_{g-i-1} \qquad i = 1, 2, \ldots, g - 2$$
$$c_0 = \ell_{g-1}$$

Proof: This result follows directly from (1.4).

3. THE DETERMINATION OF ALL DECADIC ONE-CYCLES

If one fixes $g = 10$, then each one-cycle $D = (d_n, d_{n-1}, \ldots, d_2, d_1)$ falls into one of four classes. These classes can be described using the difference counters ℓ_i, $i = 0, 1, 2, \ldots, g - 1$ introduced in Definition 2.1. The following conditions on the difference counters must hold for $D = (d_n, d_{n-1}, \ldots, d_1)$ to be in a given class.

Class A:
$\ell_8 = \ell_6 = \ell_4 = \ell_2 = 2\ell_0 + \delta$
$\ell_7 = \ell_5 = \ell_1$
$\ell_9 = 0$ iff $\ell_1 = 0$
ℓ_0, ℓ_1, or δ is nonzero

Class B:
$\ell_9 = \ell_1 = 0$ $\qquad$ $\ell_4 = \ell_2 = \ell_0 + \frac{\ell_8}{2}$
$\ell_7 = 2\ell_0 - \ell_8$ $\qquad$ $\delta = 0$
$\ell_6 = \ell_8 \neq 0$ $\qquad$ $\ell_8 \equiv 0 \pmod 2$

and $\ell_5 = \ell_3 = \ell_0 - \frac{\ell_8}{2}$ $\quad \ell_0 \geq \frac{\ell_8}{2}$

Class C: $\ell_6 = \ell_2 = 1$
$\ell_i = 0, \ i \neq 2, 3, 6$
$\delta = 0$

Class D: $\ell_5 = 2\ell_0 + \delta$
$\ell_i = 0, \ i \neq 0, 5$
ℓ_0 or δ is nonzero

Theorem 3.1: Let $(d_n, d_{n-1}, \ldots, d_1)$ be a decadic one-cycle with $d_n + d_j = 10$ and $\ell_0 = j - 1$. Suppose that $d_j \neq 5$ and either $j \neq 1$ or $\delta \neq 0$. Then $(d_n, d_{n-1}, \ldots, d_1)$ is in either Class A or Class B.

Proof: We wish to determine the difference counters ℓ_i, $i = 0, 1, 2, \ldots, g - 1$. To do this, we shall explore the various ways these differences can be computed from the digits in a self-producing integer. From Corollary 2.1,

$$c_9 = \ell_9 + 2\ell_0 + \delta$$
$$c_i = \ell_i + \ell_{9-i} \qquad i = 1, 2, \ldots, 8$$
$$c_0 = \ell_9$$

Certainly, $\ell_9 = \min(c_9, c_0) = c_0$, since a difference of 9 can only be obtained from the digits 9 and 0. Hence

$$\ell_8 = \min(2\ell_0 + \delta, c_1) = \min(2\ell_0 + \delta, \ell_1 + \ell_8) = \begin{cases} 2\ell_0 + \delta & \ell_1 \neq 0 \\ \ell_8 & \ell_1 = 0 \end{cases} \tag{3.1}$$

Thus the value of ℓ_8 depends on whether ℓ_1 is zero or nonzero. If $\ell_1 \neq 0$, then there are fewer 9's than 1's remaining and hence there will be as many differences of 8 as there are 9's remaining. If $\ell_1 = 0$, then there are fewer 1's in the self-producing integer than remaining 9's, and there will be as many differences of 8 as there are 1's. This technique of evaluating the difference counters is used throughout this section.

Suppose first that $\ell_1 \neq 0$. Note that if $\ell_1 \neq 0$, $d_j = 1$, and hence $d_n = 9$. Then we have

$$\begin{aligned} \ell_9 &= \ell_9 \neq 0 \\ \ell_8 &= 2\ell_0 + \delta \\ \ell_7 &= \ell_1 \end{aligned} \tag{3.2}$$

and
$$\ell_6 = \min(2\ell_0 + \delta, \ell_2 + \ell_7).$$

Now if $\ell_2 + \ell_7 < 2\ell_0 + \delta$, then one finds either

$$\begin{aligned} \ell_6 &= \ell_2 + \ell_7 \\ \ell_5 &= \ell_8 - (\ell_2 + \ell_7) \\ \ell_4 &= \ell_7 + \ell_2 \\ \ell_3 &= \ell_3 + \ell_6 - \ell_8 \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} \ell_6 &= \ell_2 + \ell_7 \\ \ell_5 &= \ell_3 + \ell_6 \\ \ell_4 &= \ell_8 - (\ell_2 + \ell_7 + \ell_3 + \ell_6) \\ \ell_3 &= \ell_7 + \ell_2 \\ \ell_2 &= \ell_3 \end{aligned} \tag{3.4}$$

and
$$\ell_1 = \min(\ell_2 + \ell_7, \ell_8 - \ell_2 - \ell_7)$$

Equations (3.3) imply that $\ell_6 = \ell_8$ or $\ell_2 + \ell_7 = 2\ell_0 + \delta$, which is a contradiction. Equations (3.4) imply that $\ell_1 = 0$, again a contradiction. Thus we must have $2\ell_0 + \delta \leq \ell_2 + \ell_7$. Continuing in like fashion,

$$\begin{aligned} \ell_6 &= 2\ell_0 + \delta \\ \ell_5 &= \ell_2 + \ell_7 - (2\ell_0 + \delta) \\ \ell_4 &= 2\ell_0 + \delta \\ \ell_3 &= \ell_3 \\ \ell_2 &= 2\ell_0 + \delta \\ \ell_1 &= \ell_5 \\ \ell_0 &= \frac{\ell_4 - \delta}{2} \end{aligned} \tag{3.5}$$

Equations (3.5) together with equations (3.2) determine the relations given in Class A with ℓ_1 and ℓ_9 nonzero.

Suppose now that $\ell_1 = 0$. From (3.1),

$$\begin{aligned} \ell_8 &= \ell_8 \\ \ell_7 &= \min(2\ell_0 + \delta - \ell_8,\ \ell_2 + \ell_7), \text{ or} \\ \ell_7 &= \begin{cases} 2\ell_0 + \delta - \ell_8 & \ell_2 \neq 0 \\ \ell_7 & \ell_2 = 0 \end{cases} \end{aligned} \tag{3.6}$$

We first consider the case where $\ell_2 \neq 0$. From (3.1) and (3.6) it is clear that

$$\begin{aligned} \ell_9 &= 0 \\ \ell_8 &= \ell_8 \\ \ell_7 &= 2\ell_0 + \delta - \ell_8 \\ \ell_6 &= \min(\ell_8,\ \ell_2) \end{aligned} \tag{3.7}$$

If $\ell_2 < \ell_8$, then

$$\begin{aligned} \ell_6 &= \ell_2 \\ \ell_5 &= \ell_8 - \ell_2 \\ \ell_4 &= \ell_2 + \ell_7 \\ \ell_3 &= \ell_3 + \ell_6 - \ell_8 - \ell_7 \\ \ell_2 &= \ell_3 = 2\ell_0 + \delta \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} \ell_6 &= \ell_2 \\ \ell_5 &= \ell_8 - \ell_2 \\ \ell_4 &= \ell_2 + \ell_3 + \ell_6 - \ell_8 \\ \ell_3 &= 2\ell_0 + \delta - \ell_3 - \ell_6 \\ \ell_2 &= \ell_6 + \ell_3 = \ell_2 + \ell_3 \end{aligned} \tag{3.9}$$

or

$$\begin{aligned} \ell_6 &= \ell_2 \\ \ell_5 &= \ell_3 + \ell_6 \\ \ell_4 &= \ell_8 - \ell_2 - \ell_3 - \ell_6 \\ \ell_3 &= \ell_7 + \ell_2 \\ \ell_2 &= \ell_3 + \ell_6 \end{aligned} \tag{3.10}$$

In (3.8), $\ell_5 = -\ell_7 = 0$, so $\ell_2 = \ell_8$. In (3.9), $\ell_3 = 0$, which implies $\ell_2 = \ell_8$. In (3.10), $\ell_2 = 0$, so all three circumstances lead to a contradiction. Hence, it must be that $\ell_8 \leq \ell_2$, and, therefore, in (3.7) one finds $\ell_6 = \ell_8$. In this case, there are two possible values for ℓ_4, viz., $\ell_4 = \ell_6 + \min(\ell_7, \ell_3)$.

If $\ell_4 = \ell_6 + \ell_7$, then

$$\begin{aligned} \ell_6 &= \ell_8 & \qquad \ell_2 &= \ell_6 = \ell_6 + \ell_5 \\ \ell_5 &= \ell_2 - \ell_8 & \qquad \ell_1 &= \ell_5 \\ \ell_4 &= \ell_6 + \ell_7 & \qquad \ell_0 &= \frac{\ell_6 - \delta}{2} \\ \ell_3 &= \ell_3 - \ell_7 & & \end{aligned} \tag{3.11}$$

This implies that $\ell_5 = \ell_7 = 0$, $\ell_8 = \ell_6 = \ell_4 = \ell_2 = 2\ell_0 + \delta$, so (3.11) falls into Class A with $\ell_1 = 0$. Otherwise,

$$\begin{aligned} \ell_6 &= \ell_8 & \ell_2 &= \ell_3 + \ell_6 \\ \ell_5 &= \ell_2 - \ell_8 & \ell_1 &= 0 \\ \ell_4 &= \ell_6 + \ell_3 & & \\ \ell_3 &= \ell_7 - \ell_3 & \ell_0 &= \frac{\ell_4 + \ell_5 - \delta}{2} \end{aligned} \tag{3.12}$$

Equations (3.12) fall into Class B.

It can easily be checked that there exist no one-cycles with $d_n = 7$ and $d_j = 3$ or $d_n = 6$ and $d_j = 4$. This completes the proof of the theorem.

Theorem 3.2: Let $D = (d_n, d_{n-1}, \ldots, d_1)$ be a decadic one-cycle with $d_n = 9$, $d_1 = 1$ and $\delta = 0$. Then

$$\ell_7 = \ell_5 = \ell_1 \neq 0$$
$$\ell_8 = \ell_6 = \ell_4 = \ell_2 = \ell_0 = 0,$$

and this one-cycle falls into Class A.

Proof: This results immediately from Corollary 2.1, since $\ell_0 = \delta = 0$.

Theorem 3.3: Let $D = (d_n, d_{n-1}, \ldots, d_1)$ be a decadic one-cycle with $d_n + 2d_1 = 10$ and $\delta = 0$. Then

$$\ell_6 = \ell_2 = 1$$
$$\ell_i = 0, \; i \neq 2, 3, 6;$$

hence, this one-cycle will fall into Class C.

Proof: If $d_1 = 1$, one obtains the following system of inconsistent equations:

$$\begin{aligned} \ell_8 &= 1 \\ \ell_7 &= \ell_1 - 1 \\ \ell_6 &= 1 \\ \ell_5 &= \ell_7 + \ell_2 \\ \ell_4 &= 0 \\ \ell_3 &= \ell_3 + \ell_6 = \ell_3 + 1 \end{aligned}$$

If $d_1 = 2$, then

$$\begin{aligned} \ell_6 &= 1 \\ \ell_5 &= \ell_2 - 1 \\ \ell_4 &= 0 \\ \ell_3 &= \ell_3 \\ \ell_2 &= 1 \end{aligned}$$

which falls into Class C. It can easily be checked that $d_1 = 3$ implies that $\ell_3 = \ell_3 - 1$, so the proof is complete.

Since Class D consists of all the remaining one-cycles, namely, those with $d_j = 5$ from Theorem 3.1, this completes the classification of all dedadic one-cycles.

4. THE DETERMINATION OF KAPREKAR CONSTANTS

An r-digit Kaprekar constant is an r-digit, self-producing integer such that repeated iterations of T applied to *any* starting value a will always lead to this integer. Utilizing the results of Section 3, one can now show that only for $r = 3$ and $r = 4$ does such an integer exist.

Lemma 4.1: For $r = 2n$ with $n \geq 3$, there exist at least two distinct one-cycles.

Proof: If $r = 6$, then one finds the one cycles

$$D_1 = (6, 3, 2) \text{ and } D_2 = (5, 5, 0).$$

If $r = 2n$, $n \geq 4$, then two distinct one-cycles are

$D_1: \; \ell_6 = \ell_2 = 1; \ell_3 = n - 2; \ell_i = 0, i \neq 2, 3, 6$

$D_2: \; \ell_7 = \ell_5 = \ell_1 = 1; \ell_9 = n - 3; \ell_i = 0, i \neq 1, 5, 7, 9.$

Lemma 4.2: For $r = 2n + 1$ with $n \geq 7$, there exist at least two distinct one-cycles.

Proof: If $n = 7$, then one finds the one-cycles:

$$D_1 = (8, 6, 4, 3, 3, 3, 2) \quad \text{and} \quad D_2 = (5, 5, 5, 5, 5, 0, 0).$$

If $r = 2n + 1$, $n \geq 8$, then two distinct one-cycles are:

$D_1: \ell_8 = \ell_7 = \ell_6 = \ell_5 = \ell_4 = \ell_2 = \ell_1 = 1; \ell_9 = n - 7; \ell_3 = \ell_0 = 0$

$D_2: \ell_8 = \ell_6 = \ell_4 = \ell_2 = 1; \ell_3 = n - 4; \ell_9 = \ell_7 = \ell_5 = \ell_1 = \ell_0 = 0.$

Lemma 4.3: If $r = 2, 5, 7, 9, 11$, or 13, then there does not exist a Kaprekar constant.

Proof: When $r = 2, 5$, and 7 there are no one-cycles. When $r = 9$ there are two distinct one-cycles:

$$D_1 = (5, 5, 5, 0) \quad \text{and} \quad D_2 = (8, 6, 4, 2).$$

If $r = 11$ the only one-cycle is $D_1 = (8, 6, 4, 3, 2)$, but there is also a cycle of length four, viz.,

$$(8, 8, 4, 3, 2) \to (8, 6, 5, 4, 2) \to (8, 6, 4, 2, 1) \to (9, 6, 6, 4, 2).$$

If $r = 13$ the only one-cycle is $D_1 = (8, 6, 4, 3, 3, 2)$, but there is also a cycle of length two, viz.,

$$(8, 7, 3, 3, 2, 1) \to (9, 6, 6, 5, 4, 3).$$

Theorem 4.1: The only decadic Kaprekar constants are 495 and 6174.

Proof: This follows from Lemmas 4.1-4.3.

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